9. a) Write down $ds^2$ for a flat 2D plane using the standard polar coordinates $r$ and $\theta$.

Since

$$x = r\cos\theta \qquad y = r\sin\theta \tag{1.1}$$

we get

$$\begin{aligned} dx &= dr\cos\theta - r\sin\theta\, d\theta, \\ dy &= dr\sin\theta + r\cos\theta\, d\theta \end{aligned} \tag{1.2}$$

Hence

$$\begin{aligned} ds^2 &= dx^2 + dy^2 \\ &= (\cos\theta)^2 dr^2 + r^2(\sin\theta)^2 d\theta^2 - \cancel{2r\sin\theta\cos\theta\, dr\, d\theta} \\ &\quad + (\sin\theta)^2 dr^2 + r^2(\cos\theta)^2 d\theta^2 + \cancel{2r\sin\theta\cos\theta\, dr\, d\theta} \\ &\boxed{= dr^2 + r^2 d\theta^2.} \end{aligned} \tag{1.3}$$

b) What are the coefficients of $g_{ij}$?

The metric is generally $ds^2 = g_{ij}\,dx^i dx^j$, so comparing this equation with (1.3), and identifying $dr = dx^r$, $d\theta = dx^\theta$, we get

$$(g_{ij}) \boxed{= \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}.} \tag{1.4}$$

c) Calculate the coefficients of $g^{ij}$.

Since $g_{ij}g^{jk} = \delta_i{}^k$, we need only find the inverse matrix of (1.4). We compute

$$(g^{ij}) = \frac{1}{\det g}\begin{pmatrix} r^2 & 0 \\ 0 & 1 \end{pmatrix} \boxed{= \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{r^2} \end{pmatrix}.} \tag{1.5}$$

As a check,

$$(g^{ij})(g_{jk}) = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{r^2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2/r^2 \end{pmatrix} = \mathbb{1}. \qquad (1.6)$$

d) Using

$$\Gamma^{\lambda}{}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}\left(g_{\rho\mu,\nu} + g_{\rho\nu,\mu} - g_{\mu\nu,\rho}\right) \qquad (1.7)$$

calculate the coefficients of the Christoffel symbols for this system.

Note $g_{ij}$ is constant except $g_{\theta\theta}$. Therefore $g_{ij,k} = 0$ except $i=j=\theta$. So the only derivatives that matter are

$$g_{\theta\theta,r} = \partial_r r^2 = 2r, \quad g_{\theta\theta,\theta} = 0. \qquad (1.8)$$

Furthermore $g^{\theta r} = g^{r\theta} = 0$. Clearly then

$$\boxed{\Gamma^{r}{}_{rr} = \Gamma^{\theta}{}_{\theta\theta} = 0} \qquad (1.9)$$

since the only way to introduce an $r$ or $\theta$ would be through the summation in (1.7) with coefficients $g^{\theta r} = g^{r\theta}$. Next

$$\Gamma^{r}{}_{\theta r} = \Gamma^{r}{}_{r\theta} = \frac{1}{2} g^{ri}\left(g_{ir,\theta} + g_{i\theta,r} - g_{r\theta,i}\right)$$

$$= \frac{1}{2} g^{rr}\left(g_{\cancel{rr},\theta} + g_{\cancel{r\theta},r} - g_{\cancel{r\theta},r}\right) \qquad (1.10)$$

so

$$\boxed{\Gamma^{r}{}_{\theta r} = \Gamma^{r}{}_{r\theta} = 0.} \qquad (1.11)$$

Similarly

$$\Gamma^{\theta}{}_{\theta r} = \Gamma^{\theta}{}_{r\theta} = \frac{1}{2} g^{\theta\theta}\left(\cancel{g_{\theta r,\theta}} + g_{\theta\theta,r} - \cancel{g_{r\theta,\theta}}\right)$$

$$= \frac{1}{\cancel{2}} \frac{1}{r^2}\left(\cancel{2}r\right), \qquad (1.12)$$

which gives

$$\boxed{\Gamma^{\theta}{}_{\theta r} = \Gamma^{\theta}{}_{r\theta} = \frac{1}{r}} \qquad (1.13)$$

Clearly

$$\boxed{\Gamma^{\theta}{}_{rr} = 0} \qquad (1.14)$$

because we need a second $\theta$, which could only be introduced through $g^{r\theta}$ or $g^{\theta r}$ if these were nonzero. Finally

$$\Gamma^{r}{}_{\theta\theta} = \frac{1}{2} g^{rr}\left[\cancel{g_{r\theta,\theta}} + \cancel{g_{r\theta,\theta}} - g_{\theta\theta,r}\right], \qquad (1.15)$$

which implies

$$\boxed{\Gamma^{r}{}_{\theta\theta} = -r.} \qquad (1.16)$$

e) Using

$$\ddot{x}^{\lambda} + \Gamma^{\lambda}{}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = 0 \qquad (1.17)$$

derive the two coupled, second-order differential equations that describe the motion of a free particle in this system.

The only nonzero Christoffel symbols are

$$\Gamma^{\theta}{}_{\theta r} = \Gamma^{\theta}{}_{r\theta} = \frac{1}{r}, \quad \Gamma^{r}{}_{\theta\theta} = -r. \qquad (1.18)$$

So the equations of motion become

$$0 = \ddot{x}^r + \Gamma^r{}_{\theta\theta}\dot{x}^\theta \dot{x}^\theta$$

$$= \ddot{x}^r - r\dot{x}^\theta\dot{x}^\theta$$

$$\Rightarrow \boxed{\ddot{x}^r = r\dot{x}^\theta\dot{x}^\theta} \qquad (1.19)$$

and

$$0 = \ddot{x}^\theta + 2\Gamma^\theta{}_{\theta r}\dot{x}^\theta\dot{x}^r$$

$$= \ddot{x}^\theta + \frac{2}{r}\dot{x}^\theta\dot{x}^r$$

$$\Rightarrow \boxed{\ddot{x}^\theta = -\frac{2}{r}\dot{x}^\theta\dot{x}^r.} \qquad (1.20)$$

In case it wasn't clear, my convention was $x^r = r$ and $x^\theta = \theta$. So we can rewrite these

$$\boxed{\ddot{r} = r\dot{\theta}^2, \quad \ddot{\theta} = -\frac{2}{r}\dot{r}\dot{\theta}} \qquad (1.21)$$

As a check, the units make sense.