Special and General Relativity (PHZ 4601/560 Fall 2017) Solutions Midterm October 20.

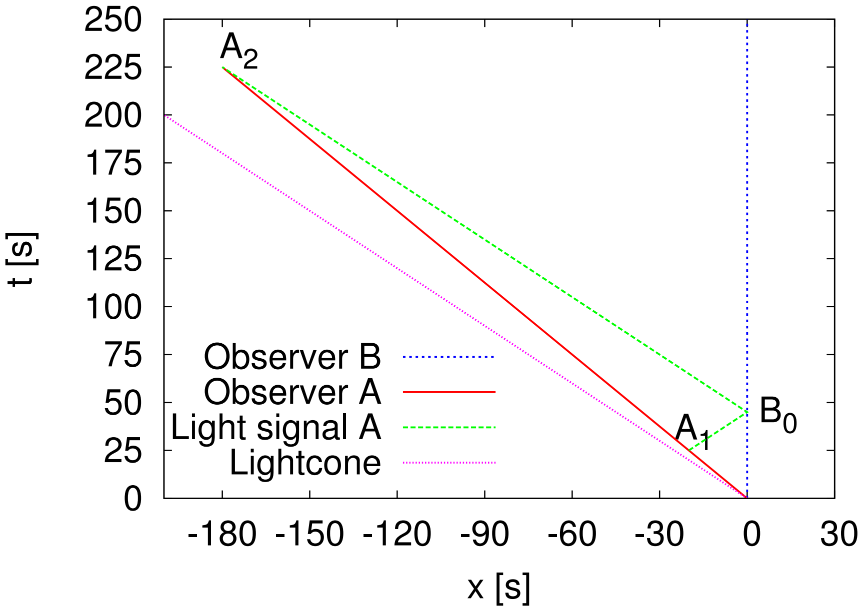

FIG. 1: Minkowski space in which observer B is at rest and observer A moves with speed $4c/5$ in negative x direction.

1. **Light signals and travel in two inertial frames.**

We use natural units, $c = 1$, and express everything in seconds.

(a) In the rest frame S of A the coordinates of A_1 are $(t, x) = (15, 0)$. In the rest frame S' of B we have $15 = \tau = \sqrt{t'^2 - x'^2}$ for the proper time of A at this space-time point and $x' = -4t'/5$. Putting these equations together, we find:

$$15 = \sqrt{(t')^2 - (4t'/5)^2} = 3\,t'/5 \quad \Rightarrow \quad t' = 25 \quad \Rightarrow \quad (t', x') = (25, -20)\,.$$

(b) In S the coordinates of B_0 are the meeting point of the straight line $x = 4t/5$ and $x = t - 15$ for $t \geq 15$. The solution is $(t, x) = (75, 60)$ and the proper time of B at this point is $\tau = \sqrt{t^2 - x^2} = 45$. In the S' frame we have by its definition $x' = 0$ for B and the space-time point B_0 becomes $(t', x') = (45, 0)$.

(c) In S the space-time position A_2 is at $(t, x) = (135, 0)$. We can deal with this like before with $(t, x) = (15, 0)$:

$$135 = 3\,t'/5 \quad \Rightarrow \quad t' = 225\,, \quad x' = -4\,t'/5 = -180\,.$$

Alternatively, we may perform the Lorentz transformations

$$t' = \gamma\, t - \beta\gamma\, x \qquad x' = \gamma\, x - \beta\gamma\, t\,.$$

Now, $\beta = 4/5$ and $\gamma = 1/\sqrt{1-(4/5)^2} = 5/3$, $\beta\gamma = 4/3$. Hence, $(t,x) = (135,0)$ transforms into

$$t' = 5t/3 = 225\,, \quad x' = -4t/3 = -180 \quad \text{i.e.,} \quad (t',x') = (225,-180)\,.$$

(d) See the figure.

2. **Spacetrip.**

We only consider the first quarter (1 year) of the flight. The other results for t are the same due to symmetry. With $\beta = v/c$ the acceleration in the rest frame is given by

$$\frac{g}{c} = \frac{d\beta}{d\tau} = \frac{d\zeta}{d\tau}\,,$$

where ζ is the rapidity. As rapidities are additive, the following equation holds in the earth frame:

$$d\zeta(\tau) = \frac{d\zeta}{d\tau}\, d\tau = \frac{g}{c}\, d\tau\,.$$

With the initial condition $\zeta(0) = 0$ this integrates to

$$\zeta = \int_0^\zeta d\zeta' = \frac{g}{c}\int_0^\tau d\tau' = \frac{g}{c}\,\tau\,.$$

The age of the twin on earth follows from $dt' = \cosh(\zeta)\, d\tau'$:

$$\int_0^t dt' = t = \int_0^\tau \cosh[\zeta(\tau')]\, d\tau' = \int_0^\tau \cosh\left[\frac{g}{c}\,\tau'\right]\, d\tau' = \frac{c}{g}\,\sinh\left[\frac{g}{c}\,\tau\right]\,.$$

Inserting $\tau = 1\,\text{year} = 365 \times 24 \times 3600\,[s]$, $c = 3 \times 10^8\,[m/s]$ and $g = 9.81\,[m/s^2]$, we find $t = 1.187$ years for a quarter of the trip and 4.748 years for the entire trip.

Distance traveled.

Seen from earth: maximum distance $= 2\,x_1$ with

$$\begin{aligned} x_1 &= \int_0^{x_1} dx = \int_0^{t_1} v(t)\, dt = \int_0^{\tau_1} \cosh\left[\zeta(\tau)\right]\, v(\tau)\, d\tau \\ &= c\int_0^{\tau_1} \cosh\left(\frac{g\tau}{c}\right)\,\tanh\left(\frac{g\tau}{c}\right)\, d\tau = c\int_0^{\tau_1} \sinh\left(\frac{g\tau}{c}\right)\, d\tau \;=\; \frac{c^2}{g}\left[\cosh\left(\frac{g\tau_1}{c}\right) - 1\right]\,. \end{aligned}$$

Numerical values: $x_1 = 0.563$ light years, maximum distance $= 1.126$ light years.