Special and General Relativity (PHZ 4601/5606) Fall 2018

Set 6

16. Four-dimensional Levi-Civita tensor.

(1) The tensor $\epsilon^{\alpha\beta\gamma\delta}$ is zero unless $\alpha\beta\gamma\delta$ is a permutation of the numbers 1234. Therefore,

$$\epsilon_{\alpha\beta\gamma\delta} = (-1)^3 \epsilon^{\alpha\beta\gamma\delta} = -\epsilon^{\alpha\beta\gamma\delta} \,.$$

(2) We have $\epsilon_{\alpha\beta\gamma_1\delta_1}\epsilon^{\alpha\beta\gamma_2\delta_2} = 0$ unless either $\gamma_1 = \gamma_2$, $\delta_1 = \delta_2$ or $\gamma_1 = \delta_2$, $\delta_1 = \gamma_2$ or (otherwise one of the already used numbers will be repeated by α or β of the sums). Therefore,

$$\epsilon_{\alpha\beta\gamma_1\delta_1}\epsilon^{\alpha\beta\gamma_2\delta_2} = a\,\delta_{\gamma_1}^{\gamma_2}\,\delta_{\delta_1}^{\delta_2} + b\,\delta_{\gamma_1}^{\delta_2}\,\delta_{\delta_1}^{\gamma_2}$$

holds. With no summation in the permutation indices π_1 to π_4 the constants follow from

$$\epsilon_{\alpha\beta\pi_2\pi_3}\epsilon^{\alpha\beta\pi_2\pi_3} = \epsilon_{\pi_4\pi_1\pi_2\pi_3}\epsilon^{\pi_4\pi_1\pi_2\pi_3} + \epsilon_{\pi_1\pi_4\pi_2\pi_3}\epsilon^{\pi_1\pi_4\pi_2\pi_3} = -2 = a\,,$$
$$\epsilon_{\alpha\beta\pi_2\pi_3}\epsilon^{\alpha\beta\pi_3\pi_2} = \epsilon_{\pi_4\pi_1\pi_2\pi_3}\epsilon^{\pi_4\pi_1\pi_3\pi_2} + \epsilon_{\pi_1\pi_4\pi_2\pi_3}\epsilon^{\pi_1\pi_4\pi_3\pi_2} = +2 = b\,.$$

(3) We have $\epsilon_{\alpha\beta_1\gamma_1\delta_1}\epsilon^{\alpha\beta_2\gamma_2\delta_2} = 0$ unless $\beta_2\gamma_2\delta_2$ is a permutation of $\beta_1\gamma_1\delta_1$. There are six such permutations, so that the results is a sum of the form

$$\begin{aligned}\epsilon_{\alpha\beta_1\gamma_1\delta_1}\epsilon^{\alpha\beta_2\gamma_2\delta_2} &= a_1\,\delta_{\beta_1}^{\beta_2}\delta_{\gamma_1}^{\gamma_2}\delta_{\delta_1}^{\delta_2} + a_2\,\delta_{\beta_1}^{\gamma_2}\delta_{\gamma1}^{\delta_2}\delta_{\delta_1}^{\beta_2} + a_3\,\delta_{\beta_1}^{\delta_2}\delta_{\gamma_1}^{\beta_2}\delta_{\delta_1}^{\gamma_2} \\ &+ b_1\,\delta_{\beta_1}^{\beta_2}\delta_{\gamma_1}^{\delta_2}\delta_{\delta_1}^{\gamma_2} + b_2\,\delta_{\beta_1}^{\delta_2}\delta_{\gamma_1}^{\gamma_2}\delta_{\delta_1}^{\beta_2} + b_3\,\delta_{\beta_1}^{\gamma_2}\delta_{\gamma_1}^{\beta_2}\delta_{\delta_1}^{\delta_2}\,,\end{aligned}$$

where it follows from (no summation) $\epsilon_{\pi_1\pi_2\pi_3\pi_4}\epsilon^{\pi_1\pi_2\pi_3\pi_4} = -1$ that $a_i = -1$ and $b_i = 1$ holds for $i = 1, 2, 3$.