Special and General Relativity (PHZ 4601/5606) Fall 2017

Set 6

19. Dual Tensor.

From ${}^*E^{\alpha\beta} = \frac{1}{2}\,\epsilon^{\alpha\beta\gamma\delta}E_{\gamma\delta}$ we obtain for the dual tensor

$$ {}^*E^{12} = E_{34},\ {}^*E^{13} = -E_{24},\ {}^*E^{14} = E_{23}\,, $$

$$ {}^*E^{23} = E_{14},\ {}^*E^{24} = -E_{13},\ {}^*E^{34} = E_{12}\,. $$

In matrix form the dual tensor reads then

$$ \left({}^*E^{\alpha\beta}\right) = \begin{pmatrix} 0 & {}^*E^{12} & {}^*E^{13} & {}^*E^{14} \\ {}^*E^{21} & 0 & {}^*E^{23} & {}^*E^{24} \\ {}^*E^{31} & {}^*E^{32} & 0 & {}^*E^{34} \\ {}^*E^{41} & {}^*E^{42} & {}^*E^{43} & 0 \end{pmatrix} = \begin{pmatrix} 0 & E_{34} & -E_{24} & E_{23} \\ -E_{34} & 0 & E_{14} & -E_{13} \\ E_{24} & -E_{14} & 0 & E_{12} \\ -E_{23} & E_{13} & -E_{12} & 0 \end{pmatrix}. $$

Using the $E_{\mu\nu}$ equations of (7.32) this becomes

$$ \left({}^*E^{\alpha\beta}\right) = \begin{pmatrix} 0 & -e_3 & e_2 & -b_1 \\ e_3 & 0 & -e_1 & -b_2 \\ -e_2 & e_1 & 0 & -b_3 \\ b_1 & b_2 & b_3 & 0 \end{pmatrix}. $$

The equation ${}^*E^{\alpha\,4}{}_{,\alpha} = 0$ is the homogeneous Maxwell equation

$$ \nabla \cdot \vec{b} = 0\,. $$

The other three equations ${}^*E^{\alpha\,i}{}_{,\alpha} = 0,\ i = 1,2,3$ yield the homogeneous Maxwell equation

$$ \nabla \times \vec{e} + \frac{1}{c}\,\frac{\partial \vec{b}}{\partial t} = 0\,. $$