Special and General Relativity (PHZ 4601/5606) Fall 2018 Solutions Set 10

24. Christoffel Symbols.

(a)

$$ds^2 = dx^2 + dy^2 = dr^2 + r^2\, d\theta^2 \ .$$

(b)

$$(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \ .$$

(c)

$$\left(g^{ij}\right) = \begin{pmatrix} 1 & 0 \\ 0 & 1/r^2 \end{pmatrix} \ .$$

(d) We use

$$\Gamma^i_{jk} = \frac{1}{2}\, g^{im}\, \left(g_{mj,k} + g_{mk,j} - g_{jk,m}\right) \tag{1.1}$$

to calculate the Christoffel symbols. Note that g_{ij} is constant except for $g_{\theta\theta}$. Therefore, $g_{ij,k} = 0$ except for

$$g_{\theta\theta,r} = 2\, r \ .$$

In the following we use the summation convention for i, j, k and m, but not for θ and r.

Inspecting Eq. (1.1) for $g_{\theta\theta,r}$ contributions, we find

$$\frac{1}{2}\, g^{\theta\, m}\, g_{m\theta,r} = \frac{1}{2}\, g^{\theta\theta}\, g_{\theta\theta,r} = \frac{1}{r}$$
$$\frac{1}{2}\, g^{r\, m}\, g_{\theta\theta,m} = \frac{1}{2}\, g^{r\, r}\, g_{\theta\theta,r} = r \ .$$

Therefore,

$$\Gamma^{\theta}_{\theta\, r} = \Gamma^{\theta}_{r\,\theta} = \frac{1}{r} \ \text{ and } \ \Gamma^{r}_{\theta\theta} = r \ .$$

All other Γ^i_{jk} are zero.

(e) So the equations of motion

$$\ddot{x}^i + \Gamma^i_{jk}\, \dot{x}^j\, \dot{x}^k = 0$$

become

$$\ddot{r} + \Gamma^r_{\theta\theta}\,\dot{\theta}\,\dot{\theta} = \ddot{r} + r\,\dot{\theta}\,\dot{\theta} = 0$$

and

$$\ddot{\theta} + \Gamma^{\theta}_{\theta\,r}\,\dot{\theta}\,\dot{r} + \Gamma^{\theta}_{r\,\theta}\,\dot{r}\,\dot{\theta} = \ddot{\theta} + \frac{2}{r}\,\dot{\theta}\,\dot{r} = 0\ .$$