Special and General Relativity (PHZ 4601/5606) Fall 2018 Solutions

Set 10

25. Motion in 2D plane.

We want to fulfill the differential equations

$$\ddot{r} = r\,\dot{\theta}^2 \quad \text{and} \quad \ddot{\theta} = -\frac{2}{r}\,\dot{r}\,\dot{\theta}\,.$$

The definitions of

$$r = \sqrt{y_0^2 + s^2} \quad \text{and} \quad \tan(\theta) = \frac{y_0}{s}\,,$$

imply the additional equations

$$\sin(\theta) = \frac{y_0}{r} \quad \text{and} \quad \cos(\theta) = \frac{s}{r}\,.$$

For the derivatives of r and θ we find

$$\dot{r} = \frac{s}{(y_0^2 + s^2)^{1/2}} = \frac{s}{r} = \cos(\theta)\,, \qquad \ddot{r} = -\dot{\theta}\,\sin(\theta)\,,$$

$$\frac{d}{ds}\,\tan(\theta) = \frac{\dot{\theta}}{\cos^2(\theta)} = -\frac{y_0}{s^2} = -\frac{1}{s}\,\tan(\theta) \;\Rightarrow\; \dot{\theta} = -\frac{\cos(\theta)}{s}\,\sin(\theta) = -\frac{\sin(\theta)}{r}\,.$$

Inserting $\sin(\theta) = -r\,\dot{\theta}$ in the equation for $\ddot{r}$ we find the first differential equation $\ddot{r} = r\,\dot{\theta}^2$.

For the second derivative of θ we have

$$\ddot{\theta} = \dot{r}\,\frac{\sin(\theta)}{r^2} - \frac{\dot{\theta}\,\cos(\theta)}{r}\,.$$

Inserting $\sin(\theta) = -r\,\dot{\theta}$ and $\cos(\theta) = \dot{r}$ we find the second differential equation:

$$\ddot{\theta} = -\frac{\dot{r}\,\dot{\theta}}{r} - \frac{\dot{r}\,\dot{\theta}}{r} = -\frac{2\,\dot{r}\,\dot{\theta}}{r}\,.$$

The motion $x = s$, $y = y_0$ in the x-y plane is described by this solution.

Quite generally the solutions are straight lines in 2D Euclidean space. So $\theta = \theta_0 =$ constant and $r = r_0 + s$ is a solution, where $r = r_0 + s$, $r_0 =$ constant follows from $\ddot{r} = 0$