

Special and General Relativity (PHZ 4601/560 Fall 2017)

Solutions Final December 14

1. Schwarzschild effective potential.

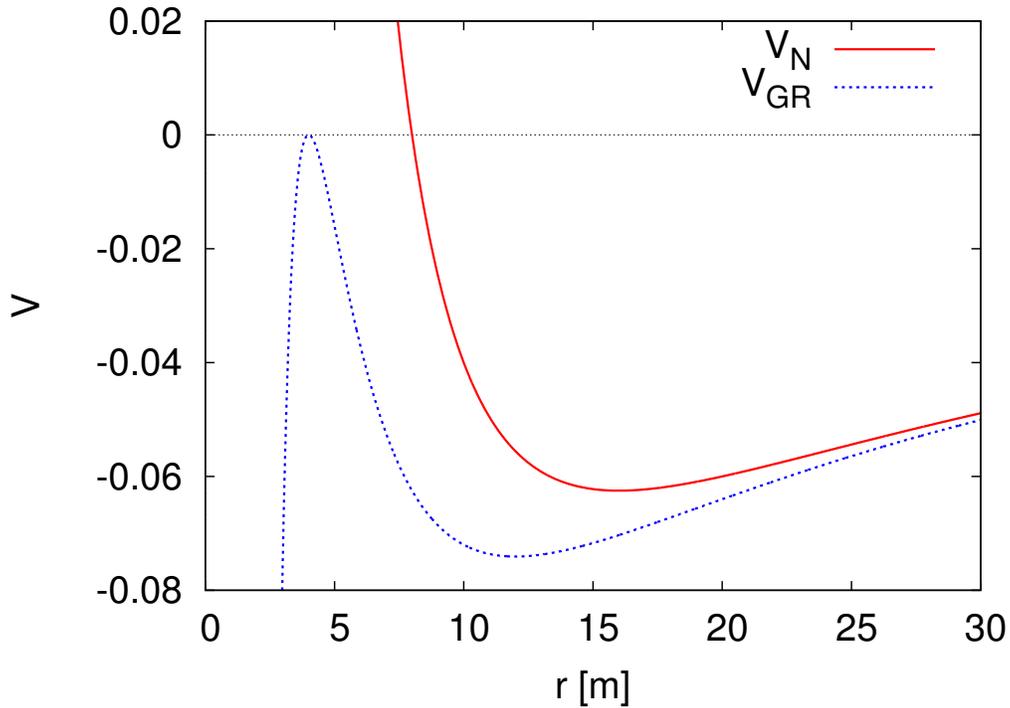


FIG. 1: Newton and Schwarzschild potentials.

(a) We have to calculate the zeros of the derivative of the effective potential

$$V'_{GR}(r) = -2 \frac{h^2}{r^3} + \frac{2m}{r^2} + \frac{6m h^2}{r^4}.$$

After multiplying with r^4 we arrive at

$$0 = -2 h^2 r + 2m r^2 + 6m h^2 \Rightarrow 0 = r^2 - \frac{h^2}{m} r + 3h^2.$$

So, $p = h^2/m$, $q = 3h^2$ and

$$r_{1,2} = \frac{h^2}{2m} \pm \sqrt{\left(\frac{h^2}{2m}\right)^2 - 3h^2}.$$

(b) In terms of h and m the equation $q = (p/2)^2 - (p/4)^2 = 3p^2/16$ reads

$$3h^2 = \frac{3h^4}{16m^2} \Rightarrow 16m^2 = h^2 \Rightarrow h = 4m,$$

resulting in $r_{1,2} = (8 \pm 4)m$ and the Schwarzschild effective potential

$$V_{GR}(r) = \frac{16m^2}{r^2} - \frac{2m}{r} - \frac{32m^3}{r^3}$$

where the first two terms are Newton's potential.

(c) For the chosen parameters the Scharzschild and Newton's potential are drawn in the figure on page 1.

2. Schwarzschild radius and density.

(a) The Scharzschild radius is $r_S = 2m$ and

$$m = \frac{4\pi}{3} r^3 \rho.$$

Therefore

$$R_0 = \frac{8\pi}{3} R_0^3 \rho \Rightarrow R_0^2 = \frac{3}{8\pi \rho}$$

and $r_S > R$ for $R > R_0$, because r_s increases $\sim R^3$, i.e., faster than R .

(b) In conventional units we have $m = GM/c^2$ and, therefore,

$$R_0^2 = \frac{3c^2}{8\pi G \rho}.$$

With the given numbers, and rounding to three digits, we find $R_0 = 4 \times 10^{27} m = 42.3 \times 10^9 ly$. This is larger than the size of the visible universe, which is approximately $15 \times 10^9 [ly]$.

3. Einstein's equations.

(a) We have

$$R^\mu{}_\mu - \frac{1}{2} g^\mu{}_\mu R = -\kappa T^\mu{}_\mu \Rightarrow R - 2R = -\kappa T \Rightarrow R = \kappa T$$

and, therefore,

$$R_{\mu\nu} - \kappa \frac{1}{2} g_{\mu\nu} T = -\kappa T_{\mu\nu} \Rightarrow R_{\mu\nu} = -\kappa \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right).$$

(b) In vacuum we have

$$R^\mu{}_\mu - \frac{1}{2} g^\mu{}_\mu R + g^\mu{}_\mu \Lambda = 0 \Rightarrow R - 2R + 4\Lambda = 0 \Rightarrow R = 4\Lambda,$$

i.e., $n = 4$.

4. Lorenz gauge with covariant derivatives.

According to (10.55) of Rindler we have

$$\Phi^\mu{}_{;\rho\sigma} - \Phi^\mu{}_{;\sigma\rho} = -\Phi^\alpha R^\mu{}_{\alpha\rho\sigma}.$$

Therefore, the definition (10.68) of the Ricci tensor yields

$$\Phi^\mu{}_{;\rho\mu} - \Phi^\mu{}_{;\mu\rho} = -\Phi^\alpha R^\mu{}_{\alpha\rho\mu} = -\Phi^\alpha R_{\alpha\rho}.$$

In the Lorenz gauge $\Phi^\mu{}_{;\mu\rho} = 0$ holds and we are left with

$$\Phi^\mu{}_{;\rho\mu} = -\Phi^\alpha R_{\alpha\rho}.$$