Lecturenotes Statistics I – Contents

Uniform and General Distributions

Uniform distribution (probability density):

$$u(x) = \begin{cases} 1 & \text{for} \quad 0 \le x < 1; \\ 0 & \text{elsewhere.} \end{cases}$$

The corresponding (cumulative) distribution function is

$$U(x) = \int_{-\infty}^{x} u(x')\, dx' = \begin{cases} 0 & \text{for} \quad x < 0; \\ x & \text{for} \quad 0 \le x \le 1; \\ 1 & \text{for} \quad x > 1. \end{cases}$$

It allows for the construction of general probability distributions. Let

$$y = F(x) = \int_{-\infty}^{x} f(x')\, dx' \; .$$

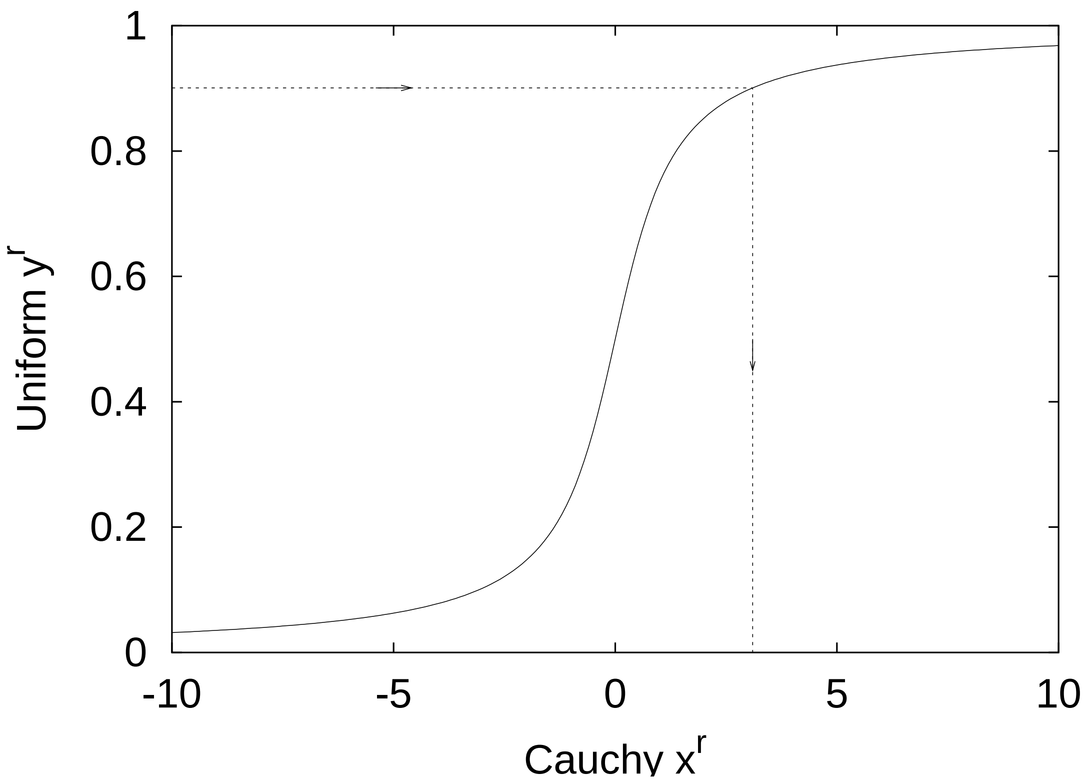
For y^r being a uniformly distributed random variable in $[0,1)$: $x^r = F^{-1}(y^r)$ is then distributed according to the probability density $f(x)$.

Example: Mapping of the uniform to the Cauchy distribution.

$$f_c(x) = \frac{\alpha}{\pi\left(\alpha^2 + x^2\right)} \quad \text{and} \quad F_c(x) = \int_{-\infty}^{x} f_c(x')\,dx' = \frac{1}{2} + \frac{1}{\pi}\,\tan^{-1}\left(\frac{x}{\alpha}\right),\ \alpha > 0\,.$$

The Cauchy distributed random variable x^r is generated from the uniform $y^r \in [0,1)$ through

$$\frac{x^r}{\alpha} = \tan\left(\pi y^r - \frac{\pi}{2}\right) \quad \Leftrightarrow \quad x^r = \alpha\,\tan(2\pi y^r)\,.$$



In STMC/Forlib: `rmacau.f`.

Gaussian Distribution

The Gaussian or **normal distribution** is of major importance. Its probability density is

$$g(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-x^2/(2\sigma^2)}$$

where σ^2 is the **variance** and $\sigma > 0$ the **standard deviation**. The Gaussian distribution function $G(x)$ is related to that of variance $\sigma^2 = 1$ by

$$G(x) = \int_{-\infty}^{x} g(x')\, dx' = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x/\sigma} e^{-(x'')^2/2}\, dx'' = \frac{1}{2} + \frac{1}{2}\,\mathrm{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)\ .$$

In principle we could now generate **Gaussian random numbers**. However, the numerical calculation of the inverse error function is slow and makes this an impractical procedure. Much faster is to express the product probability density of two independent Gaussian distributions in polar coordinates

$$\frac{1}{2\pi\,\sigma^2}\,e^{-x^2/(2\sigma^2)}\,e^{-y^2/(2\sigma^2)}\,dx\,dy = \frac{1}{2\pi\,\sigma^2}\,e^{-r^2/(2\sigma^2)}\,d\phi\,r dr\,,$$

and to use the relations $x^r = r^r\,\cos\phi^r$, $y^r = r^r\,\sin\phi^r$.

Assignments a0104_01 and a0104_02:

Probability densities and cumulative distribution functions for the uniform in $[-1,+1)$, a Gaussian and a Cauchy distribution.

Confidence Intervals and Sorting

One defines **q-tiles** (also **quantiles** or **fractiles**) x_q of a distribution function by

$$F(x_q) \;=\; q \;.$$

An example is the **median** $x_{\frac{1}{2}}$. The probability content of the **confidence interval**

$$[x_q, x_{1-q}] \;\text{ is }\; p = 1 - 2q \;.$$

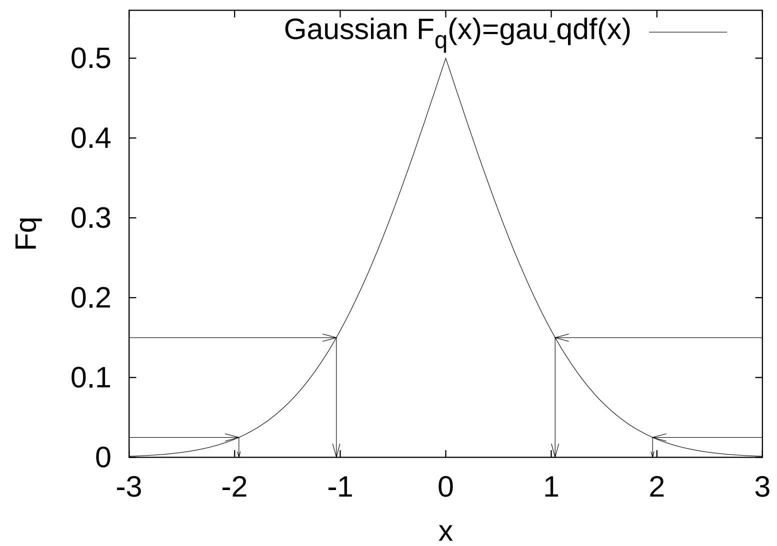
Example: Gaussian or normal distribution of variance σ^2 :

$$[-n\sigma, +n\sigma] \;\Rightarrow\; p = 0.6827 \text{ for } n = 1, \;\; p = 0.9545 \text{ for } n = 2 \;.$$

The **peaked distribution function**

$$F_q(x) = \begin{cases} F(x) \text{ for } F(x) \le \frac{1}{2}, \\ 1 - F(x) \text{ for } F(x) > \frac{1}{2}. \end{cases}$$

provides a graphical visualization of probability intervals of such a distribution:



Sorting allows for an empirical estimate. Assume we generate n random number $x_1, ..., x_n$. We may re-arrange the x_i in increasing order:

$$x_{\pi_1} \leq x_{\pi_2} \leq \ldots \leq x_{\pi_n}$$

where $\pi_1, \ldots, \pi_n$ is a permutation of $1, \ldots, n$.

An estimator for the distribution function $F(x)$ is then the **empirical distribution function**

$$\overline{F}(x) = \frac{i}{n} \quad \text{for} \quad x_{\pi_i} \le x < x_{\pi_{i+1}},\ i = 0, 1, \ldots, n-1, n$$

with the definitions $x_{\pi_0} = -\infty$ and $x_{\pi_{n+1}} = +\infty$. To calculate $\overline{F}(x)$ one needs an efficient way to **sort** n data values in ascending (or descending) order. In the STMC package this is provided by a **heapsort** routine, which arrives at the results in $O(n \log_2 n)$ steps.

Example: Gaussian distribution in assignment a0106_02 (200 and 20,000 data).