Circular Motion

$$a_r = \frac{v^2}{r}$$

The Period T

The time T required for one complete revolution is called the period. For constant speed

$$v = \frac{2\pi\, r}{T} \quad \text{holds.}$$

Example: Circular Pendulum: Figures 5-22 and 5-23 of Tipler-Mosca. Relation between angle and velocity.

$$\vec{T} + \vec{F}_{cf} + m\,\vec{g} = 0$$

$$\vec{T} = T_r\,\hat{r} + T_y\,\hat{y}$$

$$T_r = -T\,\sin(\theta) = -m\,\frac{v^2}{r}$$

$$T_y = T\,\cos(\theta) = m\,g$$

$$\frac{\sin(\theta)}{\cos(\theta)} = \frac{v^2}{g\,r}$$

$$\tan(\theta) = \frac{v^2}{g\,r}$$

$$v = \sqrt{g\,r\,\tan(\theta)}$$

Example: Forces on a car in a banked curve:

Figures 5-26 of Tipler-Mosca. The optimal angle θ is the one for which the centrifugal force is balanced by the inward component of the normal force (i.e. without friction). Then:

$$F_n \cos(\theta) - m\, g = 0$$

$$F_n \sin(\theta) - m\, \frac{v^2}{r} = 0$$

$$\tan(\theta) = \frac{v^2}{g\, r}$$

Angular Velocity

Definition:

$$\omega = \frac{d\,\theta}{d\,t}$$

For ω constant and in radian we find:

$$v = r\,\omega$$

Namely, for one period:

$$\omega\,T = 2\pi \Rightarrow v\,T = 2\pi\,r$$

Surface of Rotating Water

$$\frac{dy(r)}{dr} = \tan(\theta) = \frac{v^2}{g\,r} = \frac{\omega^2\,r^2}{g\,r} = \frac{\omega^2\,r}{g}$$

Integration:

$$y(r) = \frac{\omega\,r^2}{2\,g} + y(0)$$

parabola.

For the mathematically ambitious only:

Another Derivation of the Acceleration

Now,

$$\vec{r}(t) = r\,\hat{r} \quad \text{with} \quad \hat{r} = \cos(\theta)\,\hat{y} + \sin(\theta)\,\hat{x}$$

$$\theta(t) = \omega\, t$$

Therefore,

$$\hat{r} = \cos(\omega t)\,\hat{y} + \sin(\omega t)\,\hat{x}$$

The velocity is

$$\vec{v} = \frac{d\vec{r}}{dt} = r\,\frac{d\hat{r}}{dt} = -r\,\omega\,\sin(\omega t)\,\hat{y} + r\,\omega\,\cos(\omega\, t)\,\hat{x}$$

$$= -v\,\sin(\omega t)\,\hat{y} + v\,\cos(\omega\, t)\,\hat{x} = v\,\hat{t}$$

where

$$\hat{t} = -\sin(\omega t)\,\hat{y} + \cos(\omega\, t)\,\hat{x}$$

is the tangential unit vector. In the same way the acceleration follows:

$$\vec{a} = \frac{d\vec{v}}{dt} = v\,\frac{d\hat{t}}{dt} = -v\,\omega\,\cos(\omega t)\,\hat{y} - v\,\omega\,\sin(\omega t)\,\hat{x}$$

$$= -\frac{v^2}{r}\,\cos(\omega t)\,\hat{y} - \frac{v^2}{r}\,\sin(\omega t)\,\hat{x} = -\frac{v^2}{r}\,\hat{r}$$

$$\vec{a} = a_r\,\hat{r} \quad \text{with} \quad a_r = -\frac{v^2}{r}$$

Questions on Circular motion

A particle of mass m moves with constant speed v on a circle of radius R. The following holds (pick one):

1. The centripetal force is v^2/R towards the center.
2. The centripetal force is $m\,v^2/R$ towards the center.
3. The centripetal force is $m\,v^2/R$ away from the center.
4. The centripetal force is v^2/R away from the center.
5. The centripetal force is $m\,v^2/R$ downward.

1. The acceleration of the particle is a constant vector.
2. The acceleration of the particle is a vector of constant magnitude.
3. The magnitude of the acceleration of the particle varies with time.

1. The acceleration of the particle is a vector, which points up.
2. The acceleration of the particle is a vector, which points down.
3. The acceleration of the particle is a vector, which points towards the center of the circle.

Drag Forces

When an object moves through a gas like air or a fluid like water, it will be subject to a drag force or retarding force that reduces its speed.

For an object which falls in air under the influence of gravity one observes an acceleration like

$$m\,g - b\,v^n = m\,a$$

Where b is a constant and n is approximately one a low speed and two at high speeds.

The terminal speed v_{term} is reached for $a = 0$:

$$b\, v_{term}^n = m\, g \Rightarrow v_{term} = \left(\frac{m\, g}{b}\right)^{1/n} \quad .$$

For $n = 2$:

$$v_{term} = \sqrt{\frac{m\, g}{b}}\ .$$

Obviously, the terminal speed for a free fall in air is highly material dependent: E.g. a feather versus an iron ball, a man with our without a parachute.

Questions

$$v_{term} = \left(\frac{m\,g}{b}\right)^{1/n} \quad .$$

Determine b for an $80\,kg$ object, $n = 2$ and $v_{term} = 200\,km/h$. The result is (pick one):

1. $b = 0.254\,kg/m$

2. $b = 0.254\,kg/s$

3. $b = 0.021\,kg/s$

4. $b = 0.021\,kg/m$

Determine b for an $80\,kg$ object, $n = 1$ and $v_{term} = 20\,km/h$. The result is (pick one):

1. $b = 141\,kg/m$

2. $b = 141\,kg/s$

3. $b = 39.2\,kg/s$

4. $b = 39.2\,kg/m$