Work and Energy

Motion With Constant Force: The work W done by a constant Force $\vec{F}$ whose point of application moves through a distance $\triangle\vec{x}$ is defined to be

$$W = F\,\cos(\theta)\,\triangle x$$

where θ is the angle between the vector $\vec{F}$ and the vector $\triangle\vec{x}$, see figure 6-1 of Tipler-Mosca. If $\triangle\vec{x}$ is along the x-axis, i.e.

$$\triangle\vec{x} = \triangle x\,\hat{i} = \triangle x\,\hat{x}$$

then

$$W = F_x\,\triangle x$$

holds. Work is a scalar quantity that is positive if $\triangle x$ and F_x have the same sign and negative otherwise.

The SI unit of work and energy is the joule (J)

$$1\,J = 1\,N \cdot m = 1\,kg\,m^2\,/\,s^2$$

Another energy unit frequently used in physics is the electron volt (eV):

$$1\,eV = 1.602\,176\,462\,(63) \times 10^{-19}\,J$$

is the actual value from the National Institute of Standards and Technology (NIST), surfe physics.nist.gov. Often used multiples:

$$meV, \quad keV, \quad MeV \quad \text{and} \quad GeV\,.$$

Which of the following choices corresponds, respectively, to

$$meV, \quad keV, \quad MeV \quad \text{and} \quad GeV\,?$$

1. $10^3\,eV, \quad 10^4\,eV, \quad 10^6\,eV, \quad 10^9\,eV\,.$

2. $10^{-3}\,eV, \quad 10^2\,eV, \quad 10^3\,eV, \quad 10^6\,eV\,.$

3. $10^{-3}\,eV, \quad 10^3\,eV, \quad 10^6\,eV, \quad 10^9\,eV\,.$

4. $10^{-6}\,eV, \quad 10^3\,eV, \quad 10^6\,eV, \quad 10^9\,eV\,.$

5. $10^{-3}\,eV, \quad 10^2\,eV, \quad 10^3\,eV, \quad 10^9\,eV\,.$

6. $10^{-3}\,eV, \quad 10^2\,eV, \quad 10^3\,eV, \quad 10^6\,eV\,.$

Answer: See table 1-1 of Tipler-Mosca!

Power

The power P supplied by a force is the rate at which the force does work.

$$P = \frac{dW}{dt}$$

The SI unit of power is called watt (W):

$$1\,W = 1\,J/s$$

$$1\,kW \cdot h = (10^3\,W)\,(3600\,s) = 3.6 \times 10^6\,W \cdot s = 3.6\,MJ$$

$$1\,hp = 505\,ft \cdot lb/s = 746\,W = 0.746\,kW$$

Work and Kinetic Energy

There is and important theorem, which relates the total work done on a particle to its initial and final speeds. If F_x is the net force acting on a particle, Newton's second law gives

$$F_x = m\, a_x$$

and we recall the constant-acceleration formula (Tipler-Mosca eqn.2-17, p.28) between initial and final speeds:

$$v_f^2 - v_i^2 = 2\, a_x \triangle x \ .$$

Now, the total work becomes

$$W_{tot} = m\,a_x\,\triangle x = \frac{1}{2}\,m\,v_f^2 - \frac{1}{2}\,m\,v_i^2$$

where we substituted $a_x\,\triangle x = (v_f^2 - v_i^2)/2$. The kinetic energy of the particle is defined by:

$$K = \frac{1}{2}\,m\,v^2$$

The work-kinetic energy theorem states: The total work done on the particle is equal to the change in kinetic energy

$$W_{tot} = K_f - K_i$$

Work Done by a Variable Force

Tipler-Mosca figures 6-7 and 6-8:

$$W = \lim_{\triangle x_i \to 0} \sum_i F_x \triangle x_i = \int_{x_1}^{x_2} F_x \, dx$$

$$= \text{area under the } F_x \text{ versus } x \text{ curve.}$$

Example: Work needed to expand a spring from rest.

When we choose $x_0 = 0$ for the rest postion of the spring

$$F_x = k\,(x - x_0) = k\,x$$

Hence,

$$W = \int_0^x F_x \, dx' = \int_0^x k\,x' \, dx' = \frac{1}{2}\,k\,x^2$$

Work and Energy in 3D

Figure 6-12 of Tipler-Mosca: For a small displacement

$$\triangle W = \vec{F} \cdot \triangle \vec{s} = F \cos(\phi) \triangle s = F_s \triangle s \ .$$

Here $\vec{F} \cdot \triangle \vec{s}$ is called the dot product or scalar product of the two vectors. For two general vectors $\vec{A}$ and $\vec{B}$ it is defined by

$$\vec{A} \cdot \vec{B} = A\, B\, cos(\phi)$$

where ϕ is the angle between $\vec{A}$ and $\vec{B}$, see figure 6-13 of Tipler-Mosca.

Properties of Dot Products: Table 6-1 of Tipler-Mosca.

Commutative rule: $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$

Distributive rule: $(\vec{A} + \vec{B}) \cdot \vec{C} = \vec{A} \cdot \vec{C} + \vec{B} \cdot \vec{C}$

Further, the following holds (pick one):

1. $\vec{A}$ and $\vec{B}$ are perpendicular: $\vec{A} \cdot \vec{B} = A\,B$

2. $\vec{A}$ and $\vec{B}$ are perpendicular: $\vec{A} \cdot \vec{B} = 1$

3. $\vec{A}$ and $\vec{B}$ are perpendicular: $\vec{A} \cdot \vec{B} = 0$

1. $\vec{A}$ and $\vec{B}$ are parallel: $\vec{A} \cdot \vec{B} = A\,B$

2. $\vec{A}$ and $\vec{B}$ are parallel: $\vec{A} \cdot \vec{B} = 1$

3. $\vec{A}$ and $\vec{B}$ are parallel: $\vec{A} \cdot \vec{B} = 0$

1. $\vec{A} \cdot \vec{A} = A^2$

2. $\vec{A} \cdot \vec{A} = 1$

3. $\vec{A} \cdot \vec{A} = 0$

The General Definition of Work:

$$W = \int_{s_1}^{s_2} \vec{F} \cdot d\vec{s} = \int_{s_1}^{s_2} F_s \, ds \; .$$

3D Work–Kinetic Energy Theorem:

$$W = m \int_{s_1}^{s_2} a_s \, ds = m \int_{s_1}^{s_2} \frac{dv}{dt} \, ds = m \int_{s_1}^{s_2} \frac{dv}{ds} \frac{ds}{dt} \, ds$$

$$= m \int_{s_1}^{s_2} v \frac{dv}{ds} \, ds = m \int_{v_1}^{v_2} v \, dv = \frac{1}{2} m \, v_2^2 - \frac{1}{2} m \, v_1^2$$

Example (1): Skier skiing down a hill of constant slope.

Figure 6-18 of Tipler-Mosca:

$$W = m\,\vec{g}\cdot\vec{s} = m\,g\,s\,\cos(\phi), \quad \phi = 90^o - \theta$$

$$W = m\,g\,s\,\sin(\theta) = m\,g\,s\,\frac{h}{s} = m\,g\,h$$

Final speed v:

$$W = m\,g\,h = \frac{1}{2}\,m\,v^2 - \frac{1}{2}\,m\,v_0^2$$

where v_0 is the initial speed. For $v_0 = 0$ (initially at rest) we get for the final speed:

$$v = \sqrt{2\,g\,h}\;.$$

Example (2): Skier skiing down a hill of arbitrary slope.

Figure 6-18 of Tipler-Mosca:

$$dW = m\,\vec{g}\cdot d\vec{s} = m\,g\,ds\,\cos(\phi) = m\,g\,dh$$

$$W = \int_0^s m\,\vec{g}\cdot d\vec{s} = m\,g\int_0^h dh' = m\,g\,h$$

independently of the slope of the hill!

Potential Energy

Often work done by external forces on a system does not increase the kinetic energy of the system, but is instead stored as potential energy.

Examples (figures 6-20 and 6-21 of Tipler-Mosca):

1. Energy stored by lifting a weight.

2. Energy stored by a spring.

Conservative Forces:

A force is called conservative when its total work done on a particle along a closed path is zero (figure 6-22 of Tipler-Mosca).

Potential-Energy Function:

For conservative forces a potential energy function U can be defined, because the work done between two positions 1 and 2 does not depend on the path:

$$\triangle U = U_2 - U_1 = -\int_{s_1}^{s_2} \vec{F} \cdot d\vec{s}$$

$$dU = -\vec{F} \cdot d\vec{s} \quad \text{for infinitesimal displacements.}$$

Example: Gravitational potential energy near the earth's surface.

$$dU = -\vec{F} \cdot d\vec{s} = -(-m\, g\; \hat{j}) \cdot (dx\, \hat{i} + dy\, \hat{j} + dz\, \hat{k}) = m\, g\, dy$$

$$U = \int dU = m\, g \int_{y_0}^{y} dy' = m\, g\, y - m\, g\, y_0$$

$$U = U_0 + m\, g\, y \quad \text{with} \quad U_0 = m\, g\, y_0 \ .$$

Example: Potential energy of a spring with $x_0 = 0$.

$$dU = -\vec{F} \cdot d\vec{s} = -F_x\, dx = -(-k\, x)\, dx = k\, dx$$

$$U = \int k\, x\, dx = U_0 + \frac{1}{2}\, k\, x^2$$

We may choose $U_0 = 0$, such that U becomes

$$U = \int_0^x k\, x\, dx = \frac{1}{2}\, k\, x^2 \ .$$

Non-conservative Forces

Not all forces are conservative. Friction is an example of a non-conservative force. It eats up the energy which is converted, as we learn later, into heat.