Potential Energy and Equilibrium in 1D

Figures 6-27, 6-28 and 6-29 of Tipler-Mosca.

$$dU = -F_x\,dx$$

A particle is in equilibrium if the net force acting on it is zero:

$$F_x = -\frac{dU}{dx} = 0\ .$$

In stable equilibrium a small displacement results in a restoring force that accelerates the particle back toward its equilibirum position:

$$\frac{dF_x}{dx} = -\frac{d^2U}{dx^2} < 0 \quad \text{as} \quad F_x(x_0) = 0 \quad \text{and}$$

$$F_x(x_0 + \triangle x) = F_x(x_0) + \left.\frac{F_x}{dx}\right|_{x_0} \triangle x + O[(\triangle x)^2] \ .$$

Example: The rest position of a spring for which we have

$$\frac{dF_x}{dx} = -\frac{d^2U}{dx^2} = -k \ .$$

In unstable equilibrium a small displacement results in a force that accelerates the particle away from its equilibrium position:

$$\frac{dF_x}{dx} = -\frac{d^2U}{dx^2} > 0 \ .$$

Finally, in neutral equilibrium a small displacement results in zero force and the particle remains in equilibrium.

Questions

Assume the ball in the bowl has mass m and is positioned at a distance s s of height h above the lowest point of the bowl. Which of the following is true?

1. The potential energy of the ball is given by $m\,g\,s$.

2. The potential energy of the ball is given by $m\,g\,h$.

3. The potential energy of the bowl is given by $m\,g\,h$.

4. The potential energy of the bowl is given by $m\,g\,s$.

5. None of the above statements holds.

Combine the correct statements to one of the following choices:

A. $\int_{s_1}^{s_2} \vec{F} \cdot d\vec{s} = 0$ holds for a conservative force $\vec{F}$, where s_1 and s_2 denote the endpoints of the integration curve.

B. $\oint \vec{F} \cdot d\vec{s} = 0$ holds for a conservative force $\vec{F}$, where the integral is over a closed curve.

C. Friction is a conservative force.

D. The weight due to gravity (here close to the surface of the earth) is a conservative force.

E. The force of friction allows to define a potential.

F. A conservative force allows to define a potential.

1.ABDF, 2.BCD, 3.BCDEF, 4.ACE. 5.BDF, 6.ABCDEF, 7.ADF.

Example: The potential energy function between two atoms in a diatomic molecule is defined for $x > 0$ and given by

$$U(x) = U_0 \left[\left(\frac{a}{x} \right)^{12} - 2 \left(\frac{a}{x} \right)^{6} \right], \; U_0 > 0, \; a > 0 \; .$$

Find the equilibrium position!

$$0 = - \left. \frac{dU}{dx} \right|_{x_0} = -U_0 \left(-12 \, \frac{a^{12}}{x_0^{13}} + 12 \, \frac{a^6}{x_0^7} \right) \; .$$

$$12 \, a^{12} \, x_0^7 = 12 \, a^6 \, x_0^{13} \Rightarrow a^6 = x_0^6 \Rightarrow x_0 = a$$

Is the equilibrium stable or unstable?

$$-\left.\frac{d^2U}{dx^2}\right|_{x_0} = -U_0\left(+12\times 13\,\frac{a^{12}}{x_0^{14}} - 12\times 7\,\frac{a^6}{x_0^8}\right)$$

$$= -\frac{U_0}{a^2}\,(156-84) = -72\,\frac{U_0}{a^2} < 0 \quad \text{stable!}$$

Energy Conservation (Tipler Chapter 7)

The law of conversation of mechanical energy: For a system on which only conservative internal forces act, the sum of kinetic and potential energy is constant:

$$E = K + U = \text{constant}$$

Proof:

$$\sum_i \triangle K_i = \triangle K = W_{\text{total}} = -\sum_i \triangle U_i = -\triangle U$$

Therefore,

$$\triangle K + \triangle U = 0 \ .$$

If $E_i = K_i + U_i$ is the intitial and $E_f = K_f + U_f$ is the final mechanical energy, conservation of energy implies

$$E_f = K_f + U_f = K_i + U_i = E_i \ .$$

Example 7-2 of Tipler-Mosca: Pendulum without friction or drag forces.

$$E = K + U = \frac{1}{2} m\, v^2 + m\, g\, h$$

At the turning points:

$$E = m\, g\, h_{\max} \quad \text{as} \quad v = 0$$

At the bottom:

$$E = \frac{1}{2} m\, v_{\max}^2 \quad \text{as} \quad h = 0$$

Therefore,

$$\frac{1}{2} m\, v_{\max}^2 = m\, g\, h_{\max}$$

$$v_{\max} = \sqrt{2\, g\, h_{\max}} \quad \text{and} \quad h_{\max} = \text{constant.}$$

The conservation of mechanical energy is often much easier to use than Newton's laws for solving certain problems of mechanics.

Example: Find the speed at the bottom of the two blocks of figure 7-7 of Tipler-Mosca!

Note, the obtained results do not go beyond Newton's laws, because the conservation of mechanical energy was derived from them.

General Conservation of Energy

Let E_{sys} be the total energy of a given system, E_{in} be the energy that enters the system, E_{out} be the energy that leaves the system. The law of conservation of energy then states:

$$E_{\mathrm{in}} - E_{\mathrm{out}} = \triangle E_{\mathrm{sys}} \ .$$

Alternatively:

The total energy of the universe is constant. Energy can be converted from on form to another, or transmitted from one region to another, but energy can never be created or destroyed.

The Work-Energy Theorem: If work is the only form of energy transferred to a system, the law of conservation of energy becomes

$$W_{\mathrm{ext}} = \triangle E_{\mathrm{sys}}$$

where W_{ext} is the work done on the system by external forces.

Work-Energy Theorem with Kinetic Friction

The energy dissipated by friction is thermal energy (heat):

$$f \triangle s = \triangle E_{\mathrm{therm}}$$

where f is the frictional force applied along the distance $\triangle s$. The work-energy theorem reads then

$$W_{\mathrm{ext}} = \triangle E_{\mathrm{mech}} + \triangle E_{\mathrm{therm}} \ .$$

Other Forms of Energy

Chemical Energy: It is due to differences in the molecular binding energies. For instance, food for people and gasoline for cars. In these two cases chemical energy is released through the process of oxidation.

Nuclear Energy: Due to the nuclear binding energy, which is related to mass differences by Einstein's famous equation

$$E = m\,c^2\ .$$