Newton's Second Law for Rotation (9-4 of Tipler-Mosca)

Figure 9-16 of Tipler-Mosca shows a disk set spinning by two tangential forces $\vec{F}_1$ and $\vec{F}_2$. The points at which such forces are implied are important: The perpendicular distance between the line of action of a force and the axis of rotation is called the lever arm l of the force.

The torque τ is defined as the force times its lever arm:

$$\tau = F\, l \ .$$

PRS: In figure 9-20 of Tipler-Mosca the lever arm is (pick one):

1. $l = r\,|\sin\phi|$ or 2. $l = r\,|\cos\phi|$

where ϕ is the angle between the force $\vec{F}$ and the position vector $\vec{r}$. Therefore, the magnitude of the torque is (pick one):

1. $\tau = F\, r\,|\sin\phi|$ or 2. $\tau = F\, r\,|\cos\phi|$

In figure 9-19 of Tipler-Mosca the force $\vec{F}$ is resolved into two components: The radial force $\vec{F}_{\mathrm{r}}$ along the radial line and the tangential force $\vec{F}_{\mathrm{t}}$ perpendicular to the radial line. The torque is the given by (pick one):

1. $\tau = F_{\mathrm{r}}\, r$ or 2. $\tau = F_{\mathrm{t}}\, r$

The tangential component of the force is given by (pick one):

1. $F_{\mathrm{t}} = F\,|\sin\phi|$ or 2. $F_{\mathrm{t}} = F\,|\cos\phi|$

The torque is taken positive if it tends to rotate the disk counterclockwise, and negative if it tends to rotate the disk clockwise. Therefore, our final equation is

$$\tau = F\, r\, \sin(\phi)$$

for the situation of figures 9-19 and 9-20. The following statement holds:

The angular acceleration of a rigid body is proportional to the net torque acting on it.

Proof: Let $\vec{F}_i$ be the net external force acting on the ith particle of the rigid body. The torque on the ith particle is

$$\tau_i = F_i\, r_i\, \sin(\phi_i) = F_{\mathrm{t},i}\, r_i$$

and by Newton's second law the tangential acceleration of the ith particle is

$$F_{\mathrm{t},i} = m_i\, a_{\mathrm{t},i} = m_i\, r_i\, \alpha$$

where we use that the angular velocity and the angular acceleration are the same for all particles:

$$\frac{d\phi_i}{dt} = \omega \quad \text{and} \quad \frac{d^2\phi_i}{dt^2} = \alpha \ .$$

Therefore,

$$\tau_i = r_i \, F_{\mathrm{t},i} = m_i \, r_i^2 \, \alpha$$

and summing over all particles gives

$$\sum_i \tau_i = \left(\sum_i m_i \, r_i^2 \right) \alpha = I \, \alpha \ .$$

We thus have

$$\alpha = \frac{\tau}{I}$$

for the angular accelaration due to one torque applied to a rigid body.

Gravitational Force acting on a Wheel

The same weight of mass m pulls on two wheels, which have different moments of inertia:

$$\tau = F\,R = m\,g\,R = I\,\alpha$$

PRS and experiment: Which wheel will accelerate faster?

1. The one with the larger I.

2. The one with the smaller I.

For constant angular acceleration the time dependence of the angle is

$$\theta = \theta_0 + \omega_0\, t + \frac{1}{2}\,\alpha\, t^2$$

Assume a wheel accelerates with constant α and is initially at rest.

PRS: What is ω_0?

1. $\omega_0 = 0.$

2. $\omega_0 = \mathrm{constant} > 0.$

Assume that after 1 second the wheel has made 5 revolutions.

PRS: Assume $\theta_0 = 0$ What is, after one second, θ in radians?

1. $5\,\mathrm{rad}$ 2. 1800 rad 3. 5π rad 4. 10π rad.

PRS: What is α?

1. $10\pi\,\mathrm{rad/s^2}$ 2. $10\pi\,\mathrm{rad/s}$ 3. $20\pi\,\mathrm{rad/s^2}$ 4. $20\pi\,\mathrm{rad/s}$.

PRS: What is ω?

1. $10\pi\,\mathrm{rad/s^2}$ 2. $10\pi\,\mathrm{rad/s}$ 3. $20\pi\,\mathrm{rad/s^2}$ 4. $20\pi\,\mathrm{rad/s}$.

An Application of Newton's Second Law of Rotation

Figure 9-24 of Tipler-Mosca: An object of mass m is tied to a light string wound around a wheel that has a moment of inertia I and radius R. The wheel bearing is frictionless, and the string does not slip on the rim. Find the tension in the string and the acceleration of the object.

1. Torque on the wheel and angular acceleration:

   $$\tau = T\,R = I\,\alpha\,.$$

2. Linear acceleration of the object: $m\,g - T = m\,a\,.$

3. The non-slip condition: $a = R\,\alpha\,.$

Solving for T and a:

$$T\,R = I\,\frac{a}{R} \quad \Rightarrow \quad a = \frac{T\,R^2}{I}$$

$$m\,g - T = m\,\frac{T\,R^2}{I}$$

$$T\,\left(1 + \frac{m\,R^2}{I}\right) = m\,g$$

$$T = \frac{m\,g}{1 + m\,R^2/I} = \frac{m\,g\,I}{I + m\,R^2}$$

$$a = \frac{T\,R^2}{I} = \frac{m\,g\,R^2}{I + m\,R^2}$$

Rolling With Slipping

Example: A bowling ball of mass M and radius R is thrown so that in the instant it touches the floor it is moving with speed $v_0 = 5\,\text{m/s}$ and is not rotating. The coefficient of friction between the ball and the floor is $\mu_k = 0.08$. Find the time the ball slides before the non-slip condition is met.

1. The net force on the ball is $f_k = -\mu_k\, M\, g = M\, a_{\rm cm}$ Therefore, the linear acceleration is $a_{\rm cm} = -\mu_k\, g$

2. The linear velocity is: $v_{\rm cm} = v_0 + a_{\rm cm}\, t = v_0 - \mu_k\, g\, t$

3. The torque about the CM axis is the frictional force times the lever arm: $\tau = \mu_k \, M \, g \, R = I_{\rm cm} \, \alpha$ and with $I_{\rm cm} = 2 \, M \, R^2/5$ we get:

   $\alpha = \frac{\mu_k \, M \, g \, R}{I_{\rm cm}} = \frac{5 \, \mu_k \, M \, g \, R}{2 \, M \, R^2} = \frac{5}{2} \, \left(\frac{\mu_k \, g}{R}\right)$

4. The angular velocity is: $\omega = \alpha \, t = \frac{5}{2} \, \left(\frac{\mu_k \, g}{R}\right) \, t$

5. Solve for the time t_1 at which $v_{\rm cm} = R \, \omega$:

   $v_0 - \mu_k \, g \, t_1 = 5 \, \mu_k \, g \, t_1/2 \quad \Rightarrow \quad 2 \, v_0 = (2 + 5) \, \mu_k \, g \, t_1$

$$t_1 = \frac{2 \, v_0}{7 \mu_k \, g} = 1.82 \, \mathrm{s}$$

Cylindrical Roller

Sum of torques: (f_x force of static friction, $r_2 > r_1$)

$$F\, r_1 + f_x\, r_2 = I\, \alpha$$

CM motion :

$$F_x + f_x = M\, a_x = M\, r_2\, \alpha$$
$$F_x = F\, \cos(\theta)$$

Hollow cylinder of radius r_1 (result depends on the detailed mass-distribution):

$$F\, r_1 + f_x\, r_2 = M\, (r_1)^2\, \alpha$$
$$f_x = M\, \left(\frac{r_1}{r_2}\right)^2\, r_2\, \alpha - F\, \frac{r_1}{r_2}$$

Insert in CM motion:

$$F\left(\cos(\theta)-\frac{r_1}{r_2}\right)=\left[1-\left(\frac{r_1}{r_2}\right)^2\right]M\,a_x$$

The right-hand side factor in from of the mass is always positiv. On the left-hand side the sign of the force changes at

$$\cos(\theta)=\frac{r_1}{r_2}\ .$$