Gravity (Chapter 11)

Kepler's Laws

Example of a successful phenomenology at the cradle of science (astronomical observations by Brahe).

1. All planets move on elliptical orbits with the sun at one focus. (Ellipse: see mathworld.)

2. A line joining any planet to the sun sweeps out equal areas in equal times (figure 11-4 of Tipler-Mosca).

3. The square of the period of any planet is proportional to the cube of the semimajor axes of its orbit
$$T^2 = C\,r^2\ .$$

Astronomical unit (AU): The mean earth-sun distance

$$1\,\mathrm{AU} = 1.50 \times 10^{11}\,\mathrm{m} = 93.0 \times 10^{6}\,\mathrm{mi}\ .$$

Newton's Law of Gravity

Force of attraction between each pair of point particles (figure 11-6 of Tipler-Mosca)

$$\vec{F}_{12} = -\frac{G\, m_1\, m_2}{r_{12}^2}\, \hat{r}_{12} = -\frac{G\, m_1\, m_2}{r_{12}^3}\, \vec{r}_{12} \ .$$

Where G is the gravitational constant

$$G = 6.67 \times 10^{-11}\, \mathrm{N} \cdot \mathrm{m}^2/\mathrm{kg} \ .$$

Free-fall acceleration for objects near the surface of the earth:

$$g = \frac{G\, M_{\mathrm{E}}}{R_{\mathrm{E}}^2} = 9.81\, \mathrm{m/s}^2 \ .$$

Measurement of G: Torsion balance (first Cavendish 1798).

Gravitational and Inertial Mass

The mass, which enters the force law, is called inertial mass,

$$F = m^{\mathrm{I}}\, a \ ,$$

whereas the mass, which enters the gravitational law, is called gravitational mass,

$$\vec{F}_{12} = -\frac{G\, m_1^{\mathrm{G}}\, m_2^{\mathrm{G}}}{r_{12}^2}\, \hat{r}_{12} \ .$$

In principle, these two masses could disagree (like the electric charge has nothing to do with the mass of inertia). However, all experimental evidence is that

$$m^{\mathrm{I}} = m^{\mathrm{G}} \ .$$

Therefore, we omit the superscripts again.

Derivation of Kepler's Laws

Kepler's first law: With a bit more involved mathematics than we have presently at our disposal, one can show that the only closed solutions to Newton's two body force are elliptical orbits (intermediate mechanics for physicists).

Kepler's second law: (Figure 11-8 of Tipler-Mosca.)

The area swept out by the radius vector $\vec{r}$ in the time dt is

$$dA = \frac{1}{2}\,|\vec{r} \times \vec{v}\,dt| = \frac{1}{2\,m}\,|\vec{r} \times m\,\vec{v}|\ dt = \frac{L}{2\,m}\,dt$$

where $L = |\vec{r} \times \vec{p}|$ is the magnitude of the orbital angular momentum of the planet about the sun. Since the force on the planet is along the line from the planet to the center of the ellipse, there is no torque acting on the planet, and L is conserved. Therefore,

$$\frac{dA}{dt} = \frac{L}{2\,m} = \text{constant}\ .$$

Kepler's third law:

We will show this for the special orbit of a circle. Then

$$F = G\,\frac{m_1\,m_2}{r^2} = m_2\,a = m_2\,\frac{v^2}{r}$$

and

$$v^2 = \frac{2\pi\,r}{T} = \frac{G\,m_1}{r}$$

follows, which can be written as Kepler's third law:

$$T^2 = \frac{4\pi^2}{G\,m_1}\,r^3\ .$$

Gravitational Potential Energy

Remember, for conservative forces a potential energy is defined by

$$dU = -\vec{F}\, d\vec{s}$$

where $d\vec{s}$ is the infinitesimal displacement of the particle. For the gravitational force we have

$$dU = -\left(-G\,\frac{m_1\,m_2}{r^2}\right)\,dr = \left(G\,\frac{m_1\,m_2}{r^2}\right)\,dr$$

Integrating both sides gives

$$U = U(r) = -G\,\frac{m_1\,m_2}{r} + U_0\ .$$

In astronomy a frequently used convention for the integration constant is $U_0 = 0$, so that $U(\infty) = 0$ holds. Figure 11-9 of Tipler-Mosca gives in this convention a plot of the potential energy of a point particle in the gravitational field of the earth.

Escape Speed

Initial kinetic energy needed to escape from the earth' surface to $r = \infty$:

$$0 = K_i + U_i = \frac{1}{2} m\, v_e^2 - G\, \frac{M_{\mathrm{E}}\, m}{R_{\mathrm{E}}}$$

where v_e is called escape speed. Solving for v_e:

$$v_e = \sqrt{\frac{2\, G\, M_{\mathrm{E}}}{R_{\mathrm{E}}}} = \sqrt{2\, g\, R_{\mathrm{E}}}\ .$$

$$v_e = \sqrt{2\,(9.81\,\mathrm{m/s^2})\,(6.37 \times 10^6\,\mathrm{m})} = 11.2\,\mathrm{km/s}\ .$$

If $\quad E = K_i + U_i < 0 \quad$ the system is bound.

If $\quad E = K_i + U_i \geq 0 \quad$ the system is unbound.

Gravitational field (of the earth):

$$\vec{g} = \frac{\vec{F}}{m} = \frac{G\,M_{\rm E}}{r^2}\ .$$