Oscillations (Chapter 14)

Oscillations occur when a system is disturbed from stable equilibrium. Examples: Water waves, clock pendulum, string on musical instruments, sound waves, electric currents, ...

Simple Harmonic Motion

Example: Hooke's law for a spring.

$$F_x = m\,a = -k\,x = m\,\frac{d^2x}{dt^2}$$

$$a = \frac{d^2x}{dt^2} = -\frac{k}{m}\,x$$

The acceleration is proportional to the displacement and is oppositely directed. This defines harmonic motion.

The time it takes to make a complete oscillation is called the period T. The reciprocal of the period is the frequency

$$f = \frac{1}{T}$$

The unit of frequency is the inverse second s^{-1}, which is called a hertz Hz.

Solution of the differential equation:

$$x = x(t) = A\,\cos(\omega\,t + \delta) = A\,\sin(\omega\,t + \delta - \pi/2)$$

A, ω and δ are constants: A is the amplitude, ω the angular frequencey, and δ the phase.

$$v = v(t) = \frac{dx}{dt} = -\omega\,A\,\sin(\omega\,t + \delta)$$

$$a = a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2} = -\omega^2\,A\,\cos(\omega\,t + \delta) = -\omega^2\,x$$

Therefore, for the spring

$$\omega = \sqrt{\frac{k}{m}} \ .$$

Initial conditions: The amplitude A and the phase δ are determined by the initial position x_0 and initial velocity v_0:

$$x_0 = A \, \cos(\delta) \ \ \text{and} \ \ v_0 = -\omega \, A \, \sin(\delta) \ .$$

In particular, for the initial position $x_0 = x_{\max} = A$, the maximum displacement, we have $\delta = 0 \Rightarrow v_0 = 0$.

The period T is the time after which x repeats:

$$x(t) = x(t+T) \Rightarrow \cos(\omega\, t + \delta) = \cos(\omega\, t + \omega\, T + \delta)$$

Therefore,

$$\omega\, T = 2\pi \Rightarrow \omega = \frac{2\pi}{T} = 2\pi\, f$$

is the relationship between the frequency and the angular frequency. For Hooke's law:

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

Simple Harmonic and Circular Motion

Imagine a particle moving with constant speed v in a circle of radius $R = A$. Its angular displacement is

$$\theta = \omega\, t + \delta \quad \text{with} \quad \omega = \frac{v}{R}\, .$$

The x component of the particle's position is (figure 14-6 of Tipler-Mosca)

$$x = A\, \cos(\theta) = A\, \cos(\omega\, t + \delta)$$

which is the same as for simple harmonic motion.

Demonstration: Projected shadow of a rotating peg an an object on a spring move in unison when the periods agree.

Energy in Simple Harmonic Motion

When an objects undergoes simple harmonic motion, the systems's potential and kinetic energies vary in time. Their sum, the total energy $E = K + U$ is constant. For the force $-k\,x$, with the convention $U(x=0) = 0$, the system's potential energy is

$$U = -\int_0^x F(x')\,dx' = \int_0^x k\,x'\,dx' = \frac{k}{2}\,x^2 \ .$$

Substitution for simple harmonic motion gives

$$U = \frac{k}{2}\,A^2\,\cos^2(\omega\,t + \delta) \ .$$

The kinetic energy is

$$K = m\,\frac{v^2}{2}$$

Substitution for simple harmonic motion gives

$$K = \frac{1}{2} m\,\omega^2\,A^2\,\sin^2(\omega\,t + \delta)^2\ .$$

Using $\omega^2 = k/m$,

$$K = \frac{k}{2}\,A^2\,\sin^2(\omega\,t + \delta)^2\ .$$

The total energy is the sum

$$E = U + K = \frac{k}{2}\,A^2\,\left[\cos^2(\omega\,t + \delta) + \sin^2(\omega\,t + \delta)^2\right] = \frac{k}{2}\,A^2\ .$$

I.e., the total energy is proportional to the amplitude squared.

Plots of U and K versus t: Figures 14-7 of Tipler-Mosca.

Potential energy as function of x: Figure 14-8 of Tipler-Mosca.

Average kinetic and potential energies:

$$U_{\rm av} = K_{\rm av} = \frac{1}{2}\, E_{\rm total} \ .$$

Turning points at the maximum displacement $|x| = A$.

PRS: At the turning points the total energy is?

1. All kinetic. 2. All potential.

3. Half potential and half kinetic.

At $x = 0$ the total energy is?

1. Kinetic. 2. Potential.

3. Half potential and half kinetic.

General Motion Near Equilibrium

Any smooth potential curve $U(x)$ that has a minimum at, say x_1, can be approximated by

$$U = A + B\,(x - x_1)^2$$

and the force is given by

$$F_x = -\frac{dU}{dx} = -2B\,(x - x_1) = -k\,(x - x_1)$$

with $k = 2B$.

Compare figures 14-9 and 14-10 of Tipler-Mosca.