Work and Energy in 3D

Figure 6-11 of Tipler: For a small displacement

$$\triangle W = \vec{F} \cdot \triangle \vec{s} = F\ \cos(\phi)\, \triangle s = F_s\, \triangle s\ .$$

Here $\vec{F} \cdot \triangle \vec{s}$ is called the dot product or scalar product of the two vectors. For two general vectors $\vec{A}$ and $\vec{B}$ it is defined by

$$\vec{A} \cdot \vec{B} = A\, B\, cos(\phi)$$

where ϕ is the angle between $\vec{A}$ and $\vec{B}$, see figure 6-12 of Tipler.

Properties of Dot Products: Table 6-1 of Tipler.

Commutative rule: $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$

Distributive rule: $(\vec{A} + \vec{B}) \cdot \vec{C} = \vec{A} \cdot \vec{C} + \vec{B} \cdot \vec{C}$

Further, the following holds (pick one):

1. $\vec{A}$ and $\vec{B}$ are perpendicular: $\vec{A} \cdot \vec{B} = A\, B$
2. $\vec{A}$ and $\vec{B}$ are perpendicular: $\vec{A} \cdot \vec{B} = 1$
3. $\vec{A}$ and $\vec{B}$ are perpendicular: $\vec{A} \cdot \vec{B} = 0$

1. $\vec{A}$ and $\vec{B}$ are parallel: $\vec{A} \cdot \vec{B} = A\, B$
2. $\vec{A}$ and $\vec{B}$ are parallel: $\vec{A} \cdot \vec{B} = 1$
3. $\vec{A}$ and $\vec{B}$ are parallel: $\vec{A} \cdot \vec{B} = 0$

1. $\vec{A} \cdot \vec{A} = A^2$
2. $\vec{A} \cdot \vec{A} = 1$
3. $\vec{A} \cdot \vec{A} = 0$

The General Definition of Work:

$$W = \int_{s_1}^{s_2} \vec{F} \cdot d\vec{s} = \int_{s_1}^{s_2} F_s \, ds \; .$$

3D Work–Kinetic Energy Theorem:

$$W = m \int_{s_1}^{s_2} a_s \, ds = m \int_{s_1}^{s_2} \frac{dv}{dt} \, ds = m \int_{s_1}^{s_2} \frac{dv}{ds} \frac{ds}{dt} \, ds$$

$$= m \int_{s_1}^{s_2} v \, \frac{dv}{ds} \, ds = m \int_{v_1}^{v_2} v \, dv = \frac{1}{2} m \, v_2^2 - \frac{1}{2} m \, v_1^2$$

Example (1): Skier skiing down a hill of constant slope.

Figure 6-15 of Tipler:

$$W = m\,\vec{g}\cdot\vec{s} = m\,g\,s\,\cos(\phi), \quad \phi = 90^o - \theta$$

$$W = m\,g\,s\,\sin(\theta) = m\,g\,s\,\frac{h}{s} = m\,g\,h$$

Final speed v:

$$W = m\,g\,h = \frac{1}{2}\,m\,v^2 - \frac{1}{2}\,m\,v_0^2$$

where v_0 is the initial speed. For $v_0 = 0$ (initially at rest) we get for the final speed:

$$v = \sqrt{2\,g\,m\,h}\ .$$

Example (2): Skier skiing down a hill of arbitrary slope.

Figure 6-16 of Tipler:

$$dW = m\,\vec{g}\cdot d\vec{s} = m\,g\,ds\,\cos(\phi) = m\,g\,dh$$

$$W = \int_0^s m\,\vec{g}\cdot d\vec{s} = m\,g\int_0^h dh' = m\,g\,h$$

independently of the slope of the hill!

Potential Energy

Often work done by external forces on a system does not increase the kinetic energy of the system, but is instead stored as potential energy.

Examples (figures 6-20 and 6-21 of Tipler):

1. Energy stored by lifting a weight.
2. Energy stored by a spring.

Conservative Forces:

A force is called conservative when its total work done on a particle along a closed path is zero (figure 6-22 of Tipler).

Potential-Energy Function:

For conservative forces a potential energy function U can be defined, because the work done between two positions 1 and 2 does not depend on the path:

$$\triangle U = U_2 - U_1 = -\int_{s_1}^{s_2} \vec{F} \cdot d\vec{s}$$

$$dU = -\vec{F} \cdot d\vec{s} \ \text{ for infinitesimal displacements.}$$

Example: Gravitational potential energy near the earth's surface.

$$dU = -\vec{F}\cdot d\vec{s} = -(-m\,g\;\hat{j})\cdot(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}) = m\,g\,dy$$

$$U = \int dU = m\,g \int_{y_0}^{y} dy' = m\,g\,y - m\,g\,y_0$$
$$U = U_0 + m\,g\,y \ \ \text{with} \ \ U_0 - m\,g\,y_0 \ .$$

Example: Potential energy of a spring with $x_0 = 0$.

$$dU = -\vec{F}\cdot d\vec{s} = -F_x\,dx = -(-k\,x)\,dx = k\,dx$$

$$U = \int k\,x\,dx = U_0 + \frac{1}{2}\,k\,x^2$$

We may choose $U_0 = 0$, such that U becomes

$$U = \int_0^x k\,x\,dx = \frac{1}{2}\,k\,x^2 \ .$$

Non-conservative Forces: Not all forces are conservative. Friction is an example of a non-conservative force. It eats up the energy which is converted, as we learn later, into heat.

Potential Energy and Equilibrium in 1D

Figures 6-26, 6-27 and 6-28 of Tipler.

$$dU = -F_x \, dx$$

A particle is in equilibrium if the net force acting on it is zero:

$$F_x = -\frac{dU}{dx} = 0 \ .$$

In stable equilibrium a small displacement results in a restoring force that accelerates the particle back toward its equilibirum position:

$$\frac{dF_x}{dx} = -\frac{d^2U}{dx^2} < 0 \quad \text{as} \quad F_x(x_0) = 0 \quad \text{and}$$

$$F_x(x_0 + \triangle x) = F_x(x_0) + \left.\frac{F_x}{dx}\right|_{x_0} \triangle x + O[(\triangle x)^2] \ .$$

Example: The rest position of a spring for which we have

$$\frac{dF_x}{dx} = -\frac{d^2U}{dx^2} = -k \ .$$

In unstable equilibrium a small displacement results in a force that accelerates the particle away from its equilibrium position:

$$\frac{dF_x}{dx} = -\frac{d^2U}{dx^2} > 0 \ .$$

Finally, in neutral equilibrium a small displacement results in zero force and the particle remains in equilibrium.