Kinetic Energy of a System

Although the total momentum of a system is conserved when the net external force is zero, the total mechanical energy change, because the internal forces may be non-conservative. This is, for instance, the case in our example of the astronaut. Her muscles use chemical energy to push the pannel away. Before the push the kinetic energy of the system was zero, after the push it is

$$K_{\rm after} = m_a \frac{v_a^2}{2} + m_p \frac{v_p^2}{2} \ .$$

However, it is possible to decompose the kinetic energy of a system into a CM energy, which does not change when the net external force is zero, and kinetic energies relative to the CM, which may change due to internal forces.

The kinetic energy of a system of particles is

$$K = \sum_{i=1}^{n} K_i = \sum_{i=1}^{n} \frac{1}{2} m_i \, v_i^2 = \sum_{i=1}^{n} \frac{1}{2} m_i \, (\vec{v}_i \cdot \vec{v}_i)$$

The velocity of each particle can be written as the sum of the velocity of the CM, $\vec{v}_{\rm cm}$, and the velocity of the particle relative to the CM reference frame:

$$\vec{u}_i = \vec{v}_i - \vec{v}_{\rm cm} \quad \text{with} \quad \vec{v}_{\rm cm} = \frac{\sum_{i=1}^{n} m_i \, \vec{v}_i}{\sum_{i=1}^{n} m_i}$$

see figure 8-39 of Tipler for $i = 1, 2$.

Then

$$\vec{v}_i = \vec{v}_{\rm cm} + \vec{u}_i$$

and

$$\sum_{i=1}^{n} \frac{1}{2} m_i \, (\vec{v}_i \cdot \vec{v}_i) = \sum_{i=1}^{n} \frac{1}{2} m_i \, (\vec{v}_{\rm cm} + \vec{u}_i) \cdot (\vec{v}_{\rm cm} + \vec{u}_i)$$

$$= \sum_{i=1}^{n} \frac{1}{2} m_i \, v_{\rm cm}^2 + \sum_{i=1}^{n} \frac{1}{2} m_i \, u_i^2 + \vec{v}_{\rm cm} \cdot \sum_{i=1}^{n} m_i \, \vec{u}_i$$

Now

$$\sum_{i=1}^{n} m_i \, \vec{u}_i = M \, \vec{u}_{\rm cm} = 0$$

because the CM velocity is zero relative to the CM.

Therefore, the kinetic energy of a system of particles is

$$K = \frac{1}{2} M \, v_{\rm cm}^2 + K_{\rm rel} \quad \text{where} \quad K_{\rm rel} = \sum_{i=1}^{n} \frac{1}{2} m_i \, u_i^2 \; .$$

M is the total mass and $K_{\rm rel}$ is the kinetic energy of the particles relative to the CM.

Collisions

In a collision two objects interact strongly for a very short time, such that external forces can be neglected. When the total kinetic energy of the two objects is the same after the collision as before, the collision is called elastic. Otherwise, it is called inelastic. In the perfectly inelastic collistion all the energy relative to the center of mass is converted to thermal or internal energy of the system and the two objects stick together.

Impulse and Average Force: Figure 8-24 of Tipler.

The impulse of the force $\vec{F}$ on one of the particles is defined as

$$\vec{I} = \int_{t_\mathrm{i}}^{t_\mathrm{f}} \vec{F}\, dt$$

where the integration is over the time interval

$$\triangle t = t_\mathrm{f} - t_\mathrm{i}$$

of the collision. The magnitude of the impulse is the area under the F-versus-t curve. The impulse equals the total change in momentum during the time interval

$$\vec{I} = \int_{t_\mathrm{i}}^{t_\mathrm{f}} \frac{d\vec{p}}{dt}\, dt = \vec{p}_\mathrm{f} - \vec{p}_\mathrm{i} = \triangle \vec{p}\ .$$

The average force for the time interval is

$$\vec{F}_\mathrm{av} = \frac{1}{\triangle t} \int_{t_\mathrm{i}}^{t_\mathrm{f}} \vec{F}\, dt = \frac{\vec{I}}{\triangle t}\ .$$

Example: Figure 8-25 of Tipler.

A car with an 80 kg crash test dummy drives into a wall at 25 m/s (about 56 mi/h). Estimate the force on the seat belt.

Steps:

1. The dummy's initial momentum is $m\,v = 2000\,\mathrm{kg} \cdot \mathrm{m/s}$.
2. The impulse is the change of the momentum $I = 2000\,\mathrm{N} \cdot \mathrm{s}$.
3. Estimate the collision time. With $\triangle x = 1$ m and $v_{\mathrm{av}} = 12.5\,m/s$ one gets $\triangle t = 0.08\,s$.
4. Compute the average force:

$$F_{\mathrm{av}} = \frac{I}{\triangle t} = 25\,\mathrm{kN}\ .$$

Collisions in 1D

Momentum conservation:

$$p_{1\mathrm{f}} + p_{2\mathrm{f}} = m_1\, v_{1\mathrm{f}} + m_2\, v_{2\mathrm{f}} = m_1\, v_{1\mathrm{i}} + m_2\, v_{2\mathrm{i}} = p_{1\mathrm{i}} + p_{2\mathrm{i}}$$

Perfectly Inelastic Collisions:

$$v_{1\mathrm{f}} = v_{2\mathrm{f}} = v_{\mathrm{cm}} \quad \text{with} \quad v_{\mathrm{cm}} = \frac{m_1\, v_{1\mathrm{i}} + m_2\, v_{2\mathrm{i}}}{m_1 + m_2}$$

Example: Figure 8-29 of Tipler.

A bullet is fired into a hanging target, which is at rest. The target, with the bullet embedded, swings upward. Find the speed of the bullet from the height reached.

Solution: Energy conservation gives

$$\frac{1}{2}\,(m_1 + m_2)\, v_{\mathrm{f}}^2 = (m_1 + m_2)\, g\, h \quad \Rightarrow \quad v_{\mathrm{f}} = \sqrt{2\, g\, h}$$

Let m_1 be the mass of the bullet and m_2 be the mass of the target. Now,

$$v_{\mathrm{f}} = v_{\mathrm{cm}} = \frac{m_1\, v_{\mathrm{i}} + m_2\, 0}{m_1 + m_2}$$

and

$$v_{\mathrm{i}} = \frac{(m_1 + m_2)\,\sqrt{2\, g\, h}}{m_1}$$

is the initial speed of the bullet.

Elastic Collisions:

The final kinetic energies are equal:

$$\frac{1}{2}\, m_1\, v_{1\mathrm{f}}^2 + \frac{1}{2}\, m_2\, v_{2\mathrm{f}}^2 = \frac{1}{2}\, m_1\, v_{1\mathrm{i}}^2 + \frac{1}{2}\, m_2\, v_{2\mathrm{i}}^2$$

Rearranging this equation gives

$$m_2\, (v_{2\mathrm{f}}^2 - v_{2\mathrm{i}}^2) = m_1\, (v_{1\mathrm{i}}^2 - v_{1\mathrm{f}}^2)$$

$$m_2\, (v_{2\mathrm{f}} - v_{2\mathrm{i}})\, (v_{2\mathrm{f}} + v_{2\mathrm{i}}) = m_1\, (v_{1\mathrm{i}} - v_{1\mathrm{f}})\, (v_{1\mathrm{i}} + v_{1\mathrm{f}})$$

Momentum conservation may be re-written as

$$m_2\, (v_{2\mathrm{f}} - v_{2\mathrm{i}}) = m_1\, (v_{1\mathrm{i}} - v_{1\mathrm{f}})$$

Hence,

$$v_{2\mathrm{f}} + v_{2\mathrm{i}} = v_{1\mathrm{i}} + v_{1\mathrm{f}}$$

$$v_{2\mathrm{f}} - v_{1\mathrm{f}} = -(v_{2\mathrm{i}} - v_{1\mathrm{i}})$$

The right-hand-side of this equation is called speed of approach and the left-hand-side speed of recession.

Example: Find $v_{1\mathrm{f}}$ and $v_{2\mathrm{f}}$ for an elastic collission with

$$m_1 = m_2 = m, \quad v_{1\mathrm{i}} > 0 \ \text{ and } \ v_{2\mathrm{i}} = 0 \ .$$

Momentum conservation gives

$$v_{1\mathrm{f}} + v_{2\mathrm{f}} = v_{1\mathrm{i}}$$

and the elastic equation becomes

$$v_{2\mathrm{f}} - v_{1\mathrm{f}} = v_{1\mathrm{i}}$$

Adding both equations, we find

$$2\, v_{2\mathrm{f}} = 2\, v_{1\mathrm{i}} \quad \text{i.e.} \quad v_{2\mathrm{f}} = v_{1\mathrm{i}} \ .$$

Rocket Propulsion (mathematically ambitious)

We consider the simple model of a rocket where the fuel is burned at a constant rate

$$R = \frac{dm}{dt} = \text{constant}$$

and the speed of the exhaust gas relative to the rocket is another constant: $u_{\rm ex} > 0$. The mass of the rocket at time t becomes

$$m = m(t) = m_0 - R\,t$$

where m_0 is the rocket mass at the initial time $t = 0$. The momentum of the rocket at time t is

$$P = m\,v\ .$$

We assume that the rocket moves against a constant gravitational acceleration. In the instantaneous rest frame of the rocket momentum conservation reads

$$dP = -m\,g\,dt = m\,dv - u_{\rm ex}\,dm = m\,dv - u_{\rm ex}\,R\,dt\ .$$

Note, $dP = 0$ if there is no external force, i.e. $g = 0$. We re-write the momentum conservation equation as

$$-(m_0 - R\,t)\,g\,dt = (m_0 - R\,t)\,dv - u_{\rm ex}\,R\,dt\ .$$

Dividing both sides by $(m_0 - R\,t)$ gives

$$-g\,dt = dv - \frac{u_{\rm ex}\,R}{m_0 - R\,t}\,dt\ .$$

Such an equation is solved by the method of separation of variables. This simply means that the term with v is brought to one side of the equation and all terms with t to the other:

$$dv = -g\,dt + \frac{u_{\rm ex}\,R}{m_0 - R\,t}\,dt\ .$$

Now, both sides can be integrated:

$$\int_0^v dv' = -g\int_0^t dt' + u_{\rm ex}\,R\int_0^t \frac{1}{m_0 - R\,t'}\,dt'$$

and the integration is elementary

$$v = v(t) = -g\,t + u_{\rm ex}\,R\left[\frac{-\ln(m_0 - R\,t')}{R}\right]_0^t =$$

$$-g\,t - u_{\rm ex}\,[\ln(m_0 - R\,t) - \ln(m_0)] = -g\,t - u_{\rm ex}\,\ln\left[\frac{m}{m_0}\right]$$

which is equation (8-42) of Tipler (we used $m = m_0 - R\,t$ and the addition/subtraction rule for the logarithmic function in the last step).