Rotation (Chapter 9 of Tipler)

Figure 9-2 of Tipler shows a disk, which rotates about a fixed axis perpendicular to the disk and through its center.

Rigid body: As the disk turns, the distance between any two particles that make up the disk remains fixed.

Consider a particle at distance r from the center and let θ be the angle measures counterclockwise from a fixed reference line. When the disk rotates through and angular displacement $d\theta$, measured in radians (rad), the particle moves through a circular arc of length

$$ds = r\,|d\theta|\ .$$

The time rate of change of the angle is the angular velocity

$$\omega = \frac{d\theta}{dt}\ .$$

Question: A CD-ROM disk is rotating at 3000 revolutions per minute. What is the angular speed in radians per second?

1. 284 rad/s 2. 314 rad/s 3. 334 rad/s.

Angular acceleration:

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} \ .$$

Tangential velocity:

$$v_t = \frac{ds}{dt} = r\,\frac{d\theta}{dt} = r\,\omega \ .$$

Tangential acceleration:

$$a_t = \frac{dv_t}{dt} = r\,\frac{d\omega}{dt} = r\,\alpha \ .$$

Do not confuse this with the centripetal acceleration

$$a_c = \frac{v_t^2}{r} = \frac{(r\omega)^2}{r} = r\omega^2 \ .$$

Figure 9-3 of Tipler shows a disk set spinning by two tangential forces $\vec{F}_1$ and $\vec{F}_2$. The points at which such forces are implied are important: The perpendicular distance between the line of action of a force and the axis of rotation is called the lever arm l of the force.

The torque τ is defined as the force times its lever arm:

$$\tau = F\,l \ .$$

PRS: In figure 9-5 of Tipler the lever arm is (pick one):

1. $l = r\,|\sin\phi|$ or 2. $l = r\,|\cos\phi|$

where ϕ is the angle between the force $\vec{F}$ and the position vector $\vec{r}$. Therefore, the magnitude of the torque is (pick one):

1. $\tau = F\,r\,|\sin\phi|$ or 2. $\tau = F\,r\,|\cos\phi|$

In figure 9-6 of Tipler the force $\vec{F}$ is resolved into two components: The radial force $\vec{F}_{\rm r}$ along the radial line and the tangential force $\vec{F}_{\rm t}$ perpendicular to the radial line. The torque is the given by (pick one):

1. $\tau = F_{\rm r}\,r$ or 2. $\tau = F_{\rm t}\,r$

The tangential component of the force is given by (pick one):

1. $F_{\rm t} = F\,|\sin\phi|$ or 2. $F_{\rm t} = F\,|\cos\phi|$

The torque is taken positive if it tends to rotate the disk counterclockwise, and negative is it tends to rotate the disk clockwise. Therefore, our final equation is

$$\tau = F\, r\, \sin(\phi)$$

for the situation of figures 9-5 and 9-6. The following statement holds:

The angular acceleration of a rigid body is proportional to the net torque acting on it.

Proof: Let $\vec{F}_i$ be the net external force acting on the ith particle of the rigid body. The torque on the ith particle is

$$\tau_i = F_i\, r_i\, \sin(\phi_i)$$

and by Newton's second law the tangential acceleration of the ith particle is

$$F_{i\mathrm{t}} = m_i\, a_{i\mathrm{t}} = m_i\, r_i\, \alpha$$

where we use that the angular velocity and the angular acceleration are the same for all particles:

$$\frac{d\phi_i}{dt} = \omega \quad \text{and} \quad \frac{d^2\phi_i}{dt^2} = \alpha \ .$$

Therefore,

$$\tau_i = r_i\, F_{i\mathrm{t}} = m_i\, r_i^2\, \alpha$$

and summing over all particles gives

$$\sum_i \tau_i = r_i = \sum_i m_i\, r_i^2\, \alpha$$

The quantity

$$I = \sum_i m_i\, r_i^2$$

is called moment of inertia. It becomes

$$I = \int r^2\, dm$$

for a continuous object. We thus have

$$\alpha = \frac{\tau}{I}$$

for the angular accelaration due to one torque applied to a rigid body.