A Gravitational Force on Wheels

The same weight of mass m pulls on two wheels, which have different moments of inertia:

$$\tau = F\,R = m\,g\,R = I\,\alpha$$

PRS: Which wheel will accelerate faster?

1. The one with the larger I.

2. The one with the smaller I.

For constant angualr acceleration the time dependence of the angle is

$$\theta = \theta_0 + \omega_0\,t + \frac{1}{2}\,\alpha\,t^2$$

Assume a wheel accelerates with constant α and is initially at rest. PRS: What is ω_0?

1. $\omega_0 = 0$.

2. $\omega_0 = \text{constant} > 0$.

Assume that after 1 second the wheel has made 5 revolutions. PRS: Assume $\theta_0 = 0$ What is, after one second, θ in radians?

1. $5\,\text{rad}$ 2. $1800\,\text{rad}$ 3. $5\pi\,\text{rad}$ 4. $10\pi\,\text{rad}$.

PRS: What is α?

1. $10\pi\,\mathrm{rad/s^2}$ 2. $10\pi\,\mathrm{rad/s}$ 3. $20\pi\,\mathrm{rad/s^2}$ 4. $20\pi\,\mathrm{rad/s}$.

PRS: What is ω?

1. $10\pi\,\mathrm{rad/s^2}$ 2. $10\pi\,\mathrm{rad/s}$ 3. $20\pi\,\mathrm{rad/s^2}$ 4. $20\pi\,\mathrm{rad/s}$.

Calculating the Moment of Inertia

Discrete Systems of Particles:

Simply apply the definition

$$I = \sum_i m_i \, r_i^2$$

where r_i is the distance of particle i from the rotation axis.

Examples (PRS):

For figure 9-7 of Tipler the result is

$$I = n \, m \, a^2$$

where n is an integer. Push this number.

For figure 9-9 of Tipler n is another (or the same?) integer. Push this number.

Continuous Objects:

The moment of inertia is calculated by integration. Examples folow.

1. Uniform stick about an axis perpendicular to the stick and through one end (figure 9-10 of Tipler).

$$I = \int_0^L x^2\, dm = \int_0^L x^2 \frac{M}{L}\, dx =$$

pick one!

$$1.\ \ \frac{1}{3} M L^3 \qquad 2.\ \ \frac{1}{3} M L^2 \qquad 3.\ \ \frac{1}{2} M L^2 \ .$$

2. Hoop of radius R about a perpendicular axis through its center (figure 9-11 of Tipler).

$$I = \int r^2\, dm = R^2 \int dm = M R^2 \ .$$

3. Uniform disk of radius R about a perpendicular axis through its center (figure 9-12 of Tipler).

$$dm = \frac{M}{A}\, dA_r \quad \text{where} \quad A = \pi R^2 \quad \text{and} \quad A_r = \pi r^2$$

such that

$$\int_0^R dA_r = \int_0^R 2\,\pi\,r\,dr = \pi\,r^2\Big|_0^R = \pi\,R^2 = A\ .$$

Therefore,

$$dm = \frac{M}{\pi\,R^2}\,2\,\pi\,r\,dr = \frac{2\,M}{R^2}\,r\,dr$$

and

$$I = \int r^2\,dm = \frac{2\,M}{R^2}\int_0^R r^3\,dr =$$

pick one!

$$1.\ \frac{1}{4}\,M\,R^4 \qquad 2.\ \frac{1}{2}\,M\,R^4 \qquad 3.\ \frac{1}{2}\,M\,R^2 \qquad 4.\ \frac{2}{3}\,M\,R^2\ .$$

3. Uniform cylinder of radius R about its axis (figure 9-13 of Tipler). Using the result for the uniform disk, we have

$$I = \frac{1}{2}\,R^2\int dm_{\rm disk} = \frac{1}{2}\,M\,R^2\ .$$

Results for Uniform Bodies of Various Shapes are collected in Table 9-1 of Tipler!

Parallel Axis Theorem

The moments of inertia about an axis through the CM and a parallel axis a distance h away (figure 9-14 of Tipler) are related by

$$I_h = I_{\rm cm} + M\,h^2 \ .$$

This theorem simplifies often the calculation of the moment of inertia, because it allows to pick an axis for which the calculation is particularly simple.

An Application of Newton's Second Law of Rotation

Figure 9-18 of Tipler: An object of mass m is tied to a light string wound around a wheel that has a moment of inertia I and radius R. The wheel bearing is frictionless, and the string does not slip on the rim. Find the tension in the string and the acceleration of the object.

1. Torque on the wheel and angular acceleration:

   $$\tau = T\,R = I\,\alpha\,.$$

2. Linear acceleration of the object: $m\,g - T = m\,a\,.$

3. The non-slip condition: $a = R\,\alpha\,.$

Solving for T and a:

$$T\,R = I\,\frac{a}{R} \quad \Rightarrow \quad a = \frac{T\,R^2}{I}$$

$$m\,g - T = m\,\frac{T\,R^2}{I}$$

$$T\left(1 + \frac{m\,R^2}{I}\right) = m\,g$$

$$T = \frac{m\,g}{1 + m\,R^2/I} = \frac{m\,g\,I}{I + m\,R^2}$$

$$a = \frac{T\,R^2}{I} = \frac{m\,g\,R^2}{I + m\,R^2}$$

Rotational Kinetic Energy

$$K_{\rm rot} = \frac{1}{2}\sum_i m_i\, v_i^2 = \frac{1}{2}\sum_i m_i\,(r_i\,\omega)^2 = \frac{1}{2}\left(\sum_i m_i\, r_i^2\right)\,\omega^2$$

$$K_{\rm rot} = \frac{1}{2}\,I\,\omega^2$$

Power

$$dW = F\,ds = F\,r\,d\theta = \tau\,d\theta$$

$$P = \frac{dW}{dt} = \tau\,\frac{d\theta}{dt} = \tau\,\omega$$

Rolling Objects without Slipping

Figure 9-26 of Tipler. The point of contact moves a distance

$$s = R\,\phi\ .$$

The velocity and acceleration of the CM are

$$v_{\rm cm} = R\,\omega \quad \text{and} \quad a_{\rm cm} = R\,\alpha\ .$$

The kinetic energy of a rolling object is

$$K = \frac{1}{2}\,M\,v_{\rm cm}^2 + \frac{1}{2}\,I\,\omega^2\ .$$

Example: Objects rolling down an inclined plane. The terminal velocity follows from energy conservation:

$$m\,g\,h = K = m\,\frac{v_{\rm cm}^2}{2} + I\,\frac{v_{\rm cm}^2}{2\,R^2} = \left(m + \frac{I}{R^2}\right)\,\frac{v_{\rm cm}^2}{2}$$

$$v_{\rm cm}^2 = \frac{2\,m\,g\,h}{m + I/R^2}$$

where the moment of inertia I is about the CM axis.

A hoop (1), a full disk (2), and a body on tiny wheels (3) have equal masses. Starting from rest, they roll down the same inclined plane. In which order do they arrive?

1. 123 2. 132 3. 213 4. 231 5. 312 6. 321

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