Rolling With Slipping

Example: A bowling ball of mass M and radius R is thrown such that in the instant it touches the floor it is moving with speed $v_0 = 5\,\mathrm{m/s}$ and is not rotating. The coefficient of friction between the ball and the floor is $\mu_k = 0.08$. Find the time the ball slides before the non-slip condition is met.

1. The net force on the ball is $f_k = -\mu_k\, M\, g = M\, a_{\rm cm}$ Therefore, the linear acceleration is $a_{\rm cm} = -\mu_k\, g$

2. The linear velocity is: $v_{\rm cm} = v_0 + a_{\rm cm}\, t = v_0 - \mu_k\, g\, t$

3. The torque about the CM axis is the frictional force times the lever arm: $\tau = \mu_k\, M\, g\, R = I_{\rm cm}\, \alpha$ and with $I_{\rm cm} = 2\, M\, R^2/5$ we get:

   $\alpha = \frac{\mu_k\, M\, g\, R}{I_{\rm cm}} = \frac{5\,\mu_k\, M\, g\, R}{2\, M\, R^2} = \frac{5}{2}\,\left(\frac{\mu_k\, g}{R}\right)$

4. the angular velocity is: $\omega = \alpha\, t = \frac{5}{2}\,\left(\frac{\mu_k\, g}{R}\right)\, t$

5. Solve for the time t_1 at which $v_{\rm cm} = R\,\omega$:

   $v_0 - \mu_k\, g\, t_1 = 5\,\mu_k\, g\, t_1/2 \quad \Rightarrow \quad 2\, v_0 = (2+5)\,\mu_k\, g\, t_1$

$$t_1 = \frac{2\, v_0}{7\mu_k\, g} = 1.82\,\mathrm{s}$$

The Vector Nature of Rotation

1. The angular velocity $\vec{\omega}$.

   Right-hand-rule: Tipler figure 10-2.

2. In accordance with the right-hand-rule the torque is defined as a vector: Figures 10-3 and 10-4 of Tipler.

   Mathematically this is expressed by defining the torque as cross (or vector) product:

$$\vec{\tau} = \vec{r} \times \vec{F}$$

   General definition of the vector product:

$$\vec{C} = \vec{A} \times \vec{B} = A\, B\, \sin(\phi)\, \hat{n}$$

   where ϕ is the angle between the vectors and $\hat{n}$ is a unit vector that is perpendicular to $\vec{A}$ and $\vec{B}$ and in the direction given by the right-hand-rule: Tipler figure 10-5.

   Not that the magnitude C of $\vec{C}$ is the area of the parallelogram.

Properties of the cross product:

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A} \quad \Rightarrow \quad \vec{A} \times \vec{A} = 0$$

Distributive law:

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

Product rule for derivatives:

$$\frac{d}{dt}(\vec{A} \times \vec{B}) = \frac{d\vec{A}}{dt} \times \vec{B} + \vec{A} \times \frac{d\vec{B}}{dt}$$

Vector products of the unit vectors are related by cyclic permutation:

$$\hat{x} \times \hat{y} = \hat{z}, \quad \hat{y} \times \hat{z} = \hat{x}, \quad \hat{z} \times \hat{x} = \hat{y}$$

In Tipler notation:

$$\hat{i} \times \hat{j} = \hat{k}, \quad \hat{j} \times \hat{k} = \hat{i}, \quad \hat{k} \times \hat{i} = \hat{j}$$

Angular Momentum

Definition:

$$\vec{L} = \vec{r} \times \vec{p}$$

Like torque angular momentum is defined with respect to the point in space where the position vector $\vec{r}$ originates. For a rotation around a symmetry axis z we find

$$\vec{L} = I\,\vec{\omega}$$

see figures 10-10 and 10-11 of Tipler. Proof:

$$\vec{L} = R\,m\,v_1\,\hat{z} + R\,m\,v_2\,\hat{z} + r_z\,m\,v_1\,\hat{n} - r_z\,m\,v_2\,\hat{n}$$

If $\hat{z}$ is a symmetry axis $v_1 = v_2$ and:

$$\vec{L} = 2\,m\,R\,v\,\hat{z} = 2\,m\,R^2\,\vec{\omega} = I\,\vec{\omega}\ .$$

The angular momentum about any point O' is the angular momentum about the center of mass, called spin angular momentum, plus the angular momentum associated with the motion of the center of mass about O', called orbital angular momentum:

$$\vec{L} = \vec{L}_{\rm orbit} + \vec{L}_{\rm spin} = \vec{r}_{\rm cm} \times M\,\vec{v}_{\rm cm} + \sum_i \vec{r'}_i \times m_i\,\vec{u}_i$$

Torque and Angular Momentum

The net external torque acting on a system equals the rate of change of the angular momentum of the system:

$$\sum_i \vec{\tau}_{i,\mathrm{ext}} = \frac{d\vec{L}}{dt}$$

Proof:

$$\vec{\tau}_{\mathrm{net}} = \vec{r} \times \vec{F}_{\mathrm{net}} = \vec{r} \times \frac{d\vec{p}}{dt}$$

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p}) = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt}$$

and

$$\frac{d\vec{r}}{dt} \times \vec{p} = \vec{v} \times m\,\vec{v} = 0\ .$$