Conservation of Angular Momentum

If the net external torque acting on a system is zero, the total angular momentum of the system is constant.

$$\frac{d\vec{L}}{dt} = \vec{\tau}_{\text{net}} = 0 \quad \Rightarrow \quad \vec{L} = \text{constant.}$$

This has important consequences!

PRS:

Rotation on a chair. By pulling the weights closer to the body, the rotation will become (pick one):

1. slower 2. faster

Motion of a Gyroscope

We have

$$\vec{\tau}_{\rm net} = \frac{d\vec{L}}{dt} \quad \text{or} \quad \vec{\tau}_{\rm net}\, dt = d\vec{L}$$

and for the net torque

$$\vec{\tau}_{\rm net} = \vec{r}(t) \times M\, \vec{g}\ .$$

Now,

$$d\vec{L} = L\, d\hat{l} + \hat{l}\, dL$$

where $|\vec{L}|$ is the magnitude of the angular momentum and $\hat{l}$ the unit vector in the direction of the momentum. Note, Tipler on p.303 is using dL for the magnitude of $d\vec{L}$ instead for the infinitesimal change of the magnitude of $\vec{L}$. From the derivation of Tipler it remains unclear where the approximation is.

The angular momentum of the wheel around its CM symmetry (spin) axis is

$$\vec{L}_{\rm cm} = I_s\, \vec{\omega}_s\ .$$

Assume that $\vec{L}_{\rm cm}$ is very large compared to:

1. The orbital angular momentum.

2. The change (derivative) of the magnitude of the angular momentum.

Then, we can approximate the change in the angular momentum by

$$\vec{\tau}_{\rm net}\, dt = d\vec{L} = I_s\, \omega_s\, d\hat{r}$$

where $\hat{r}$ is the unit vector in the direction of the symmetry axis. This can be written as

$$D\, M\, g\, \hat{\tau}\, dt = I_s\, \omega_s\, \omega_p\, \hat{\tau}\, dt$$

where D is the distance of the wheel from the center, $\hat{\tau}$ the unit vector in the direction of the torque and ω_p the angular velocity of the unit vector $\hat{r}$ around the origin. (Note,

$$d\vec{r} = \vec{v}\, dt = r\, \vec{\omega}\, dt$$

and $r = 1$ for the unit vector.)

Due to the angular velocity

$$\omega_p = \frac{M\,g\,D}{I_s\,\omega_s}$$

we have a motion in the direction of the torque, which is called precession.

There are corrections to our approximation, in particuar due to the initial gain of orbital angular momentum. These corrections lead to an up-and-down oscillation, called nutation, of the axle.

Clutch

Tipler figure 10-21: A disk is rotating with an initial angular speed ω_1 about a frictionless symmetry axis. Its moment of inertia about this axis is I_1. It is dropped on another disk of moment of inertia I_2 about the same symmetry axis, which is initially at rest. Due to friction the two disks attain eventually a common angular speed ω_f. Find ω_f.

Angular momentum conservations gives:

$$L_f = (I_1 + I_2)\,\omega_f = L_i = I_1\,\omega_1$$

$$\omega_f = \frac{I_1\,\omega_1}{I_1 + I_2}$$

How much kinetic energy is lost?

$$K_f = (I_1 + I_2)\,\frac{\omega_f^2}{2} \quad \text{and} \quad K_i = I_1\,\frac{\omega_1^2}{2}$$

$$\frac{K_f}{K_i} = \frac{I_1}{I_2 + I_2}$$

Merry-Go-Round

Tipler figure 10-22: A merry-go-round of radius $R = 2\,\mathrm{m}$ and moment of inertia $I_m = 500\,\mathrm{kg{\cdot}m^2}$ is rotating about a frictionless pivot, making one revolution per $5\,\mathrm{s}$. A child of mass $m = 25\,\mathrm{kg}$, originally standing at the center walks out the rim. Consider the child as a point particle of mass m and find the new angular speed of the merry-go-round.

At the rim:

$$I_{\text{child}} = m\,R^2$$

Angular momentum conservation:

$$L_f = (I_m + m\,R^2)\,\omega_f = L_i = I_m\,\omega_i$$

(Initially $I_{\text{child}} = 0$ as it stays at the center.) Therefore,

$$\omega_f = \frac{I_m\,\omega_i}{I_m + m\,R^2} = \frac{500}{500 + 25 \cdot 2^2}\,\frac{1\,\mathrm{rev}}{5\,\mathrm{s}} = \frac{1\,\mathrm{rev}}{6\,\mathrm{s}}$$