Kinematics

Displacement of a point particle:

$$\triangle\vec{x} = \vec{x}_2 - \vec{x}_1$$

where $\vec{x}_1$ is the position at time t_1 and $\vec{x}_2$ is the position at time t_2, $t_2 > t_1$.

Instantaneous velocity

$$\vec{v} = \frac{d\,\vec{x}}{d\,t}$$

This is the slope of the tangent of the curve $\vec{x}(t)$ at t and called derivative.

The instantaneous acceleration is:

$$\vec{a} = \frac{d\,\vec{v}}{d\,t} = \frac{d^2\vec{x}}{dt^2}$$

Motion With Constant Acceleration

$$\frac{d\,\vec{v}}{d\,t} = \vec{a} = \vec{a}_{\text{average}}$$

Integration:

$$\vec{v} = \frac{d\,\vec{x}}{d\,t} = \vec{v}_0 + \vec{a}\,t$$

Here $\vec{v}_0$ is the velocity at time zero, the first initial condition.

Second integration:

$$\vec{x} = \vec{x}_0 + \vec{v}_0\,t + \frac{1}{2}\,\vec{a}\,t^2$$

Here $\vec{x}_0$ is the second initial condition, the position at time zero.

Newton's Laws

1. Law of inertia. An object continues to travel with constant velocity (including zero) unless acted on by an external force.

2. The acceleration $\vec{a}$ of an object is given by

$$m\,\vec{a} = \vec{F}_{\rm net} = \sum_i \vec{F}_i$$

   where m is the mass of the object and $\vec{F}_{\rm net}$ the net external force.

3. Action $=$ Reaction. Forces always occur in equal and opposite pairs. If object A exterts a force on object B, an equal but opposite force is exterted by object B on A.

Example: Two Connected Blocks

Tipler figure 5-10 (there with friction, here without friction).

PRS:

How many forces act on block 2?

How many forces act on block 1?

$$T = m_1\, a$$
$$(m_2\, g - T) = m_2\, a$$
$$m_2\, g = m_1\, a + m_2\, a = (m_1 + m_2)\, a$$
$$a = \frac{m_2\, g}{m_1 + m_2}$$

Friction

Friction is a complicated phenomenon that arises when the electromagnetic interactions of molecules between two surfaces in close contact lead to a bonding.

If an external force acts on a heavy box standing on a floor (see figure 5-1 of Tipler), the box may not move because the external force is balanced by the force f_s of static friction. Its maximum value $f_{s,max}$ is obtained when any further increase of the external force will cause the box to slide. To a good approximation $f_{s,max}$ is simply proportional to the normal force

$$f_{s,max} = \mu_s F_n$$

where μ_s is called the coefficient of static friction. If the box does not move we have

$$f_s \le f_{s,max} \ .$$

Kinetic friction (also called sliding friction): Once the box slides, a constant force is needed to keep it sliding at constant velocity. The opposing force is the force of kinetic friction. In a good approximation it is also simply proportional to the normal force

$$f_k = \mu_k \, F_n$$

where μ_k is called the coefficient of kinetic friction.

Experimentally it is found that $\mu_k < \mu_s$.

PRS: Again, the two connected blocks.

How many forces act on block 1 when we add friction?

Another example: Figure 5-48 of Tipler.

PRS:

How many forces act on block 3?

How many forces act on block 1?

How many forces act on block 2?

Let μ_k be the coefficient of kinetic friction and μ_s be the coefficient of static friction. What is the maximum acceleration of block 1?

$$1.\ \ a_{\max} = \mu_k \, g \qquad 2.\ \ a_{\max} = m_1 \, \mu_k \, g$$

$$3.\ \ a_{\max} = \mu_s \, g \qquad 4.\ \ a_{\max} = m_1 \, \mu_s \, g$$

Homework: Find two equation for the corresponding maximum tension in the string.

Work and Energy

Motion With Constant Force:

The work W done by a constant Force $\vec{F}$ whose point of application moves through a distance $\triangle\vec{x}$ is defined to be

$$W = F\,\cos(\theta)\,\triangle x$$

where θ is the angle between the vector $\vec{F}$ and the vector $\triangle\vec{x}$, see figure 6-1 of Tipler.

If $\triangle\vec{x}$ is along the x-axis, i.e.

$$\triangle\vec{x} = \triangle x\,\hat{i} = \triangle x\,\hat{x}$$

then

$$W = F_x\,\triangle x$$

holds. Work is a scalar quantity that is positive if $\triangle x$ and F_x have the same sign and negative otherwise.

The SI unit of work and energy is the joule (J)

$$1\,J = 1\,N \cdot m = 1\,kg\,m^2\,/\,s^2$$

Work and Kinetic Energy

There is and important theorem, which relates the total work done on a particle to its initial and final speeds. If $\vec{F}$ is the net force acting on a particle, Newton's second law gives

$$\vec{F} = m\,\vec{a}$$

The total work becomes

$$W_{tot} = m\,\vec{a}\,\triangle\vec{x} = \frac{1}{2}\,m\,\vec{v}_f^2 - \frac{1}{2}\,m\,\vec{v}_i^2$$

The kinetic energy of the particle is defined by:

$$K = \frac{1}{2}\,m\,\vec{v}^2$$

and the mechanical work-kinetic energy theorem states: The total work done on the particle is equal to the change in kinetic energy

$$W_{tot} = K_f - K_i$$

Potential Energy

Often work done by external forces on a system does not increase the kinetic energy of the system, but is instead stored as potential energy.

Conservative Forces:

A force is called conservative when its total work done on a particle along a closed path is zero (figure 6-22 of Tipler).

Potential-Energy Function:

For conservative forces a potential energy function U can be defined, because the work done between two positions 1 and 2 does not depend on the path:

$$\triangle U = U_2 - U_1 = -\int_{s_1}^{s_2} \vec{F} \cdot d\vec{s}$$

$dU = -\vec{F} \cdot d\vec{s}$ for infinitesimal displacements.

Example: Gravitational potential energy near the earth's surface.

$$dU = -\vec{F}\cdot d\vec{s} = -(-m\,g\,\hat{j})\cdot(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}) = m\,g\,dy$$

$$U = \int dU = m\,g \int_{y_0}^{y} dy' = m\,g\,y - m\,g\,y_0$$

PRS:

Potential energy of a simple pendulum (figure 7-2 of Tipler). Let L be the length of the weightless string and h be the height of the turning point and θ be the angle corresponding to it.

Which equation for h is correct?

1. $h = L\,\cos(\theta)$
2. $h = L\,\sin(\theta)$
3. $h = L\,[1 - \cos(\theta)]$
4. $h = L\,[1 - \sin(\theta)]$

What is the velocity of the pendulum at its lowest point?

1. $v_{\max} = \sqrt{2\,g\,h}$
2. $v_{\max} = \sqrt{g\,h}$
3. $v_{\max} = \sqrt{2\,g\,h\,(1 - \cos\theta)}$

Work-Energy Theorem with Kinetic Friction

Non-conservative Forces: Not all forces are conservative. Friction is an example of a non-conservative force. The energy dissipated by friction is thermal energy (heat):

$$f \triangle s = \triangle E_{\rm therm}$$

where f is the frictional force applied along the distance $\triangle s$. The work-energy theorem reads then

$$W_{\rm ext} = \triangle E_{\rm mech} + \triangle E_{\rm therm} \; .$$

Example: Block on an inclined plane with friction, Tipler figures 5-6 and 5-7.

PRS:

How many forces act on the block?

PRS (continued):

Assume the block (approximated as a point particle) starts from rest and slides down the inclined plane a distance L. Let v_f be its velocity at the bottom of the inclined plane. What is the change of potential energy of the block?

1. $m\,g\,L\,\cos(\theta)$ 2. $-\,m\,\frac{v_f^2}{2}$ 3. $m\,g\,L\,\sin(\theta)$

Assume the coefficient of kinetic friction is μ_k. Which is the correct equation for the frictional force?

1. $f_k = \mu_k\,m\,g\,\cos(\theta)$ 2. $f_k = \mu_k\,m\,g\,\sin(\theta)$

How much work is done by the frictional force?

1. $W_f = f_k\,L$ 2. $W_f = f_k\,L\,\sin(\theta)$

3. $W_f = f_k\,L\,\cos(\theta)$

PRS (continued):

What is the final kinetic energy $K_f = m\,v_f^2/2$ of the block at the bottom of the ramp?

1. $K_f = m\,g\,h$ with $h = L\,\sin(\theta)$
2. $K_f = m\,g\,h$ with $h = L\,\cos(\theta)$
3. $K_f = m\,g\,h - W_f$ with $h = L\,\sin(\theta)$
4. $K_f = m\,g\,h - W_f$ with $h = L\,\cos(\theta)$

Homework:

1. Find the velocity of the block at the bottom.
2. How long does it take the block to simply fall from a height h?
3. During such a fall, how far would the block travel in the horizontal direction, assuming its velocity in this direction is v_x.

Momentum Conservation

The Center of Mass (CM):

The CM $\vec{r}_{\rm cm}$ moves as if all the external foces acting on the system were acting on the total mass M of the system located at this point. In particular, it moves with constant velocity, if the external forces acting on the system add to zero.

Definition:

$$M\,\vec{r}_{\rm cm} = \sum_{i=1}^{n} m_i\,\vec{r}_i \quad \text{where} \quad M = \sum_{i=1}^{n} m_i\ .$$

Here the sum is over the particles of the system, m_i are the masses and $\vec{r}_i$ are the position vectors of the particles. In case of a continuous object, this becomes

$$M\,\vec{r}_{\rm cm} = \int \vec{r}\,dm$$

where dm is the position element of mass located at position $\vec{r}$.

Momentum Conservation

Definition: The mass of a particle times it velocity is called momentum

$$\vec{p} = m\,\vec{v}\ .$$

Newton's second law can be written as

$$\vec{F}_{\rm net} = \frac{d\vec{p}}{dt} = \frac{d(m\vec{v})}{dt} = m\,\frac{d\vec{v}}{dt} = m\,\vec{a}$$

as the masses of our particles have been constant.

The total momentum $\vec{P}$ of a system is the sum of the momenta of the individual particles:

$$\vec{P} = \sum_{i=1}^{n} \vec{p}_i = \sum_{i=1}^{n} m_i\,\vec{v}_i = M\,\vec{v}_{\rm cm}$$

Differentiating this equation with respect to time, we obtain

$$\frac{d\vec{P}}{dt} = M\,\frac{d\vec{v}_{\rm cm}}{dt} = M\,\vec{a}_{\rm cm} = \vec{F}_{\rm net,ext}$$

The law of momentum conservation: When the net external force is zero, the total momentum is constant

$$\vec{F}_{\rm net,ext} = 0 \quad \Rightarrow \quad \vec{P} = {\rm constant.}$$

Example:

Inelastic scattering, figure 8-29 of Tipler.

A bullet of mass m_1 is fired into a hanging target of mass m_2, which is at rest. The bullet gets stuck in the target. Find the speed v_i of the bullet from the joint velocity v_f of bullet and target after the collision.

PRS:

The result is

$$1.\quad v_i = \frac{m_1 + m_2}{m_1}\, v_f \qquad 2.\quad v_i = \frac{m_1 + m_2}{m_2}\, v_f$$

Solution: Momentum conservation gives

$$p_i = m_1\, v_i = (m_1 + m_2)\, v_f = p_f$$

Let $M = m_1 + m_2$. Assume, the bob is attached to a massless, rigid rod of length L. Which of the following equations gives the minimum speed v_f needed such that the bob swings through a full circle?

$$1.\quad 2\,M\,g\,L = M\,\frac{v_f^2}{2} \qquad 2.\quad M\,\frac{v_f^2}{2} = 2\,M\,g\,L + M\,\frac{g\,L}{2}$$

Rotation

1. The angular velocity $\vec{\omega}$. Direction: Right-hand-rule.

2. In accordance with the right-hand-rule the torque is defined as a vector: $\vec{\tau} = \vec{r} \times \vec{F}$.

3. Angular Momentum Definition: $\vec{L} = \vec{r} \times \vec{p}$.

   Like the torque angular momentum is defined with respect to the point in space where the position vector $\vec{r}$ originates. For a rotation around a symmetry axis we find $\vec{L} = I\,\vec{\omega}$ (magnitude $L = I\,\omega$).

4. Rotational kinetic energy: $K_{\rm rot} = \frac{1}{2} I\,\omega^2$.

Examples of moments of inertia (needed):

$I_{\rm ss} = (2/5)\,m\,R^2$ for a solid sphere.

$I_{\rm hs} = (2/3)\,m\,R^2$ for hollow sphere.

Torque and Angular Momentum

The net external torque acting on a system equals the rate of change of the angular momentum of the system:

$$\sum_i \vec{\tau}_{i,\mathrm{ext}} = \frac{d\vec{L}}{dt}$$

Conservation of Angular Momentum

If the net external torque acting on a system is zero, the total angular momentum of the system is constant.

$$\frac{d\vec{L}}{dt} = \vec{\tau}_{\mathrm{net}} = 0 \quad \Rightarrow \quad \vec{L} = \mathrm{constant}.$$

Equation Sheet

$$\vec{v} = \frac{d\vec{r}}{dt} \qquad \vec{a} = \frac{d\vec{v}}{dt} \qquad a = \frac{v^2}{R}$$

$$\vec{r} = \vec{r}_0 + \vec{v}_0\, t + \frac{1}{2}\,\vec{a}\, t^2$$

$$F = m\, a \qquad f_s = \mu_s\, F_N \qquad f_k = \mu_k\, F_N$$

$$K = \frac{1}{2}\, m\, v^2 \qquad U = m\, g\, h \qquad g = 9.81\,\mathrm{m/s^2}$$

$$F = -k\, x \qquad U = \frac{1}{2}\, k\, x^2$$

$$\triangle E = \triangle K + \triangle U \qquad F_r = -\frac{dU}{dr} \qquad P = \frac{dW}{dt} = \vec{F}\cdot\vec{v}$$

$$\vec{r}_{\rm cm} = \frac{\sum_i m_i\, \vec{r}_i}{\sum_i m_i} \qquad p = m\, v \qquad p_i = p_f$$

$$s = r\, \theta \qquad \omega = \frac{d\theta}{dt} \qquad \alpha = \frac{d\omega}{dt}$$

$$\theta = \theta_0 + \omega_o\, t + \frac{1}{2}\,\alpha\, t^2$$

$$I_P = I_{\rm cm} + M\, d^2 \qquad \vec{\tau} = \vec{r}\times\vec{F} \qquad \vec{\tau} = I\,\vec{\alpha} = \frac{d\vec{L}}{dt}$$

$$L = I\,\omega \qquad K_{\rm rot} = \frac{1}{2}\, I\,\omega^2$$

$$I_{\rm solid\ sphere} = \frac{2}{5}\, M\, R^2 \qquad I_{\rm hollow\ sphere} = \frac{2}{3}\, M\, R^2$$