Oscillations (Chapter 14)

Oscillations occur when a system is disturbed from stable equilibrium. Examples: Water waves, clock pendulum, string on musical instruments, sound waves, electric currents, ...

Simple Harmonic Motion

Example: Hooke's law for a spring.

$$F_x = m\,a = -k\,x = m\,\frac{d^2x}{dt^2}$$

$$a = \frac{d^2x}{dt^2} = -\frac{k}{m}\,x$$

The acceleration is proportional to the displacement and is oppositely directed. This defines harmonic motion.

The time it takes to make a complete oscillation is called the period T. The reciprocal of the period is the frequency

$$f = \frac{1}{T}$$

The unit of frequency is the inverse second s^{-1}, which is called a hertz Hz.

Solution of the differential equation:

$$x = x(t) = A\,\cos(\omega\,t + \delta) = A\,\sin(\omega\,t + \delta - \pi/2)$$

A, ω and δ are constants: A is the amplitude, ω the angular frequencey, and δ the phase.

$$v = v(t) = \frac{dx}{dt} = -\omega\,A\,\sin(\omega\,t + \delta)$$

$$a = a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2} = -\omega^2\,A\,\cos(\omega\,t + \delta) = -\omega^2\,x$$

Therefore, for the spring

$$\omega = \sqrt{\frac{k}{m}}\ .$$

Initial conditions: The amplitude A and the phase δ are determined by the initial position x_0 and initial velocity v_0:

$$x_0 = A\,\cos(\delta) \ \text{ and } \ v_0 = -\omega\,A\,\sin(\delta)\ .$$

In particular, for the initial position $x_0 = x_{\max} = A$, the maximum displacement, we have $\delta = 0 \Rightarrow v_0 = 0$.

The period T is the time after which x repeats:

$$x(t) = x(t + T) \Rightarrow \cos(\omega\, t + \delta) = \cos(\omega\, t + \omega\, T + \delta)$$

Therefore,

$$\omega\, T = 2\pi \Rightarrow \omega = \frac{2\pi}{T} = 2\pi\, f$$

is the relationship between the frequency and the angular frequency. For Hooke's law:

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

Simple Harmonic and Circular Motion

Imagine a particle moving with constant speed v in a circle of radius $R = A$. Its angular displacement is

$$\theta = \omega\, t + \delta \quad \text{with} \quad \omega = \frac{v}{R}\;.$$

The x component of the particle's position is (figure 14-6 of Tipler)

$$x = A\,\cos(\theta) = A\,\cos(\omega\, t + \delta)$$

which is the same as for simple harmonic motion. Demonstrations: Figure 14-7 of Tipler.

Energy in Simple Harmonic Motion

When an objects undergoes simple harmonic motion, the systems's potential and kinetic energies vary in time. Their sum, the total energy $E = K + U$ is constant. For the force $-k\,x$, with the convention $U(x=0)=0$, the system's potential energy is

$$U = -\int_0^x F(x')\,dx' = \int_0^x k\,x'\,dx' = \frac{k}{2}\,x^2 \ .$$

Substitution for simple harmonic motion gives

$$U = \frac{k}{2}\,A^2\,\cos^2(\omega\,t + \delta) \ .$$

The kinetic energy is

$$K = m\,\frac{v^2}{2}$$

Substitution for simple harmonic motion gives

$$K = \frac{1}{2} m\,\omega^2\,A^2\,\sin^2(\omega\,t + \delta)^2 \ .$$

Using $\omega^2 = k/m$,

$$K = \frac{k}{2} A^2 \omega^2 \sin^2(\omega t + \delta)^2 .$$

The total energy is the sum

$$E = U + K = \frac{k}{2} A^2 \left[\cos^2(\omega t + \delta) + \sin^2(\omega t + \delta)^2\right] = \frac{k}{2} A^2 .$$

I.e., the total energy is proportional to the amplitude squared.

Plots of U and K versus t: Figure 14-8 of Tipler.

Potential energy as function of x: Figure 14-9 of Tipler.

Average kinetic and potential energies:

$$U_{\rm av} = K_{\rm av} = \frac{1}{2} E_{\rm total} .$$

Turning points at the maximum displacement $|x| = A$.

Questions (PRS)

At the turning points the total energy is:

1. All kinetic. 2. All potential.

3. Half potential and half kinetic.

At $x = 0$ the total energy is:

1. Kinetic. 2. Potential.

3. Half potential and half kinetic.

General Motion Near Equilibrium

Any smooth potential curve $U(x)$ that has a minimum at, say x_1, can be approximated by

$$U = A + B\,(x - x_1)^2$$

and the force is given by

$$F_x = -\frac{dU}{dx} = -2B\,(x - x_1) = -k\,(x - x_1)$$

with $k = 2B$.

Example: Figure 14-11 of Tipler shows the potential energy function $U(r)$ for the separation r of two hydrogen atoms.

Examples of Oscillating Systems

Object on a Vertical Spring:

$$m\,\frac{d^2y}{dt^2} = F_y(y) = -k\,y + m\,g\ .$$

Equilibrium position:

$$0 = F_y(y_0) = -k\,y_0 + m\,g\ \Rightarrow\ y_0 = m\,g/k\ .$$

Substitution of $y = y' + y_0$ into Newton's equation gives

$$m\,\frac{d^2(y'+y_0)}{dt^2} = m\,\frac{d^2y'}{dt^2} = -k\,y' - k\,y_0 + m\,g = -k\,y'\ .$$

This is the equation of harmonic motion with the solution

$$y' = A\,\cos(\omega\,t + \delta)\ .$$

So, if we measure the displacement from the equilibrium position, we can forget about the effect of gravity (figure 14-12 of Tipler).

The Simple Pendulum: Figure 14-14 of Tipler.

$$s = L\,\phi \quad \text{where} \quad \phi \quad \text{is in radians.}$$

Newton's second law:

$$F_t = m\,\frac{d^2 s}{dt^2} = m\,L\,\frac{d^2\phi}{dt^2}\ .$$

Question (PRS): The absolute value of the tangential force is

1. $|F_t| = m\,g\,\sin(\phi)\ .$ 2. $|F_t| = m\,g\,\cos(\phi)\ .$

Removing the absolute value from F_t, the sign on the right-hand-side is:

1. positive. 2. negative.

Therefore,

$$F_t = -m\,g\,\sin(\phi) = \frac{d^2 s}{dt^2} = m\,L\,\frac{d^2\phi}{dt^2}$$

$$\frac{d^2\phi}{dt^2} = -\frac{g}{L}\,\sin(\phi)\ .$$

For small oscillations we have $\sin(\phi) \approx \phi$, and

$$\frac{d^2\phi}{dt^2} = -\frac{g}{L}\,\phi = -\omega^2\,\phi\ .$$

The period is thus

$$T = \frac{2\pi}{\omega} = 2\pi\,\sqrt{\frac{L}{g}}$$

and the solution for the motion of the angle is

$$\phi = \phi_0\,\cos(\omega\,t + \delta)$$

where ϕ_0 is the maximum angular displacement.

Pendulum in an Accelerated Reference Frame:

Figure 14-15 of Tipler.

The solution is found from the simple pendulum by replacing g with g' where

$$\vec{g'} = \vec{g} - \vec{a}$$

and $\vec{a}$ is the acceleration. As $\vec{g}$ and $\vec{a}$ are perpendicular, we have

$$g' = |\vec{g'}| = \sqrt{\vec{g}^{\,2} + \vec{a}^{\,2}} \ .$$

The Physical Pendulum: Figure 14-17 of Tipler.

This is a rigid object pivoted about a point other than its center of mass. It will oscillate when displaced from equilibrium. Newton's second law of rotation is:

$$\tau = I\,\alpha = I\,\frac{d^2\phi}{dt^2}$$

where α is the angular acceleration and I the moment of inertia about the pivot point.

Question (PRS): The torque is given by

1. $\tau = -M\,g\,D\,\sin(\phi)$.
2. $\tau = -M\,g\,D\,\cos(\phi)$.

Therefore,

$$-M\,g\,D\,\sin(\phi) = I\,\frac{d^2\phi}{dt^2}$$

$$\frac{d^2\phi}{dt^2} = -\frac{M\,g\,D}{I}\,\sin(\phi) \approx -\frac{M\,g\,D}{I}\,\phi = -\omega^2\,\phi$$

and the period is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{M\,g\,D}}\ .$$