Damped Oscillations

Left to itself, an oscillation stops eventually, because mechanical energy is dissipated by frictional forces. Such motion is said to be damped. The damping force can be represented by the empirical expression

$$\vec{F}_d = -b\,\vec{v}$$

where b is a constant. The motion of a damped system can be calculated from Newton's second law

$$F_x = -k\,x - b\,\frac{dx}{dt} = m\,\frac{d^2x}{dt^2}$$

The solution for this equation is found using standard methods (see Tipler p.428 ff.) for solving differential equations:

$$x = A_0\,e^{-(b/2m)t}\,\cos(\omega'\,t + \delta) = A_0\,e^{-t/2\tau}\,\cos(\omega'\,t + \delta)$$

where A_0 is the maximum amplitude and

$$\tau = \frac{m}{b}$$

is called decay time or time constant.

The frequency ω' is given by

$$\omega' = \omega_0 \sqrt{1 - \left(\frac{b}{2m\,\omega_0}\right)^2} \quad \text{where} \quad \omega_0 = \sqrt{\frac{k}{m}}$$

Here ω_0 is the frequency without damping. The dashed curves in figure 14-20 of Tipler correspond to $x = \pm A$ where A is given by

$$A = A_0\, e^{-(b/2m)t} = A_0\, e^{-t/\tau} \ .$$

If the damping constant b is gradually increased, we have

$$\omega' = 0 \ \text{ at the critical value } \ b_c = 2m\,\omega_o \ .$$

When $b \geq b_c$, the system does not oscillate:

$b = b_c$: The system is critically damped.

$b > b_c$: The system is overdamped.

One uses critical damping to return to equilibrium quickly. Example: Shock absorbers of a car.

$b < b_c$: The system is underdamped (often simply called damped).

Average energy of an underdamped oscillator:

$$E = \frac{1}{2} m \omega^2 A^2 = \frac{1}{2} m \omega^2 \left(A_0 \, e^{-t/2\tau} \right)^2 = E_0 \, e^{-t/\tau}$$

$$\text{where} \quad E_0 = \frac{1}{2} m \omega^2 A_0^2$$

The Q factor (for quality factor) of an oscillator relates to the fractional energy loss per cycle. The infinitesimal change of the energy is

$$dE = -\frac{1}{\tau} E_0 \, e^{-t/\tau} \, dt = -\frac{1}{\tau} E \, dt \ .$$

If the energy loss per period, $\triangle E$, is small, we can replace dE by $\triangle E$ and dt by T (also $\omega' \approx \omega_o$):

$$\frac{|\triangle E|}{E} = \frac{T}{\tau} = \frac{2\pi}{\omega_0 \, \tau} = \frac{2\pi}{Q}$$

with

$$Q = \omega_o \, \tau = \frac{\omega_o \, m}{b} = \frac{2\pi}{(|\triangle E|/E)_{\text{cycle}}} \ .$$

Driven Oscillations and Resonance

To keep a damped oscillator going, energy must be put into the system. Example: When you keep a swing going, you drive an oscillator.

The natural frequency ω_0 of an oscillator is the frequency when no driving or damping forces are present.

We assume that the oscillator is driven by a periodic motion of driving frequency ω.

When the driving frequency equals the natural frequency, the energy absorbed is at its maximum. Therefore, the natural frequency is also called the resonance frequency of the system.

In most applications the angular frequency $\omega = 2\pi f$, instead of the frequency, is used, because this is mathematically more convenient. In verbal descriptions the word “angular” is often omitted.

A resonance curve shows the average power delivered to and oscillator as function of the driving frequency: For two different damping constants the resonance curve is plotted in figure 14-25 of Tipler. The resonance is sharp for small damping.

For small damping the ratio of the width of the resonance to the frequency can be shown to be equal to the reciprocal Q factor

$$\frac{\triangle\omega}{\omega_0} = \frac{\triangle f}{f} = \frac{1}{Q} \; .$$

Intuitively, we know how to drive an oscillator at its resonance frequency (swing, etc.).