Wave Motion (Chapter 15)

Waves are moving oscillations. They transport energy and momentum through space without transporting matter. In mechanical waves this happens via a disturbance in a medium.

Transverse waves: The disturbance is perpendicular to the direction of transportation (figure 15-1).

Longitudinal waves: The disturbance is parallel to the propagation (figure 15-2).

Wave Pulses:

In a co-moving reference frame a wave pulse is at all times described by a function $f(x')$. The x coordinate in the Lab system is

$$x = x' \pm v\,t$$

where the wave pulse becomes

$$\begin{aligned} y &= f(x - v\,t) \quad \text{wave moving right} \\ \text{or} \quad y &= f(x + v\,t) \quad \text{wave moving left} . \end{aligned}$$

The function f is called wave function.

Speed of Waves:

The speed depends on the properties of the medium but is independent of the motion of the source of the waves. For example, the speed of sound from a car depends only on the properties of the air and not on the motion of the car.

For wave pulses on a string one has

$$v = \sqrt{\frac{F}{\mu}}$$

where F is the tension (T is used for the period) and μ the linear mass density.

For sound waves in a fluid the speed is

$$v = \sqrt{\frac{B}{\rho}}$$

where $B = \triangle P/(\triangle V/V)$ is defines the bulk modulus (P pressure and V volume) and ρ is the equilibrium density of the medium.

For sound waves in a gas such as air, the bulk modulus is proportional to the pressure, wich in turn is proportional to the density ρ and the absolute temperature T_K the gas (chapter 19). Then,

$$v = \sqrt{\frac{\gamma\, R\, T_K}{M}}$$

where $R = 8.314\,\mathrm{J/mol \cdot K}$ is the universal gas constant, M is the molar mass of the gas and γ is a constant, which characterizes the kind of gas ($\gamma = 1.4$ for diatomic molecules).

Derivation of v for a string:

A small segment of the string $\triangle s = R\,\theta$ is moving on a circular path (figure 15-5 of Tipler). It is subject to the radial force

$$F_r = 2\,F\,\sin(\theta/2) = F\,\theta$$

The mass of the element is $m = \mu\,\triangle s = \mu\,R\,\theta$. As v^2/R is the centripetal acceleration, Newton's second law gives

$$F_r = m\,\frac{v^2}{R} = \mu\,R\,\theta\,\frac{v^2}{R}\ .$$

Putting the two equation for F_r together, we find

$$F\,\theta = \mu\,R\,\theta\,\frac{v^2}{R} \;\Rightarrow\; v = \sqrt{\frac{F}{\mu}}\;.$$

The Wave Equation

We can apply Newton's laws to a segment of the string to derive a differential equation, known as the wave equation, which relates the spatial derivative of $y(x,t)$ to its time derivatives.

Figure 15-6 of Tipler shows a segment of the string. We consider only small vertical displacements. Then the length of the segment is approximately $\triangle x$ and its mass is $m = \mu \triangle x$, where μ is the string's mass per unit length. The segment moves vertically and the net force in this direction is

$$F_y = F\,\sin(\theta_2) - F\,\sin(\theta_1)$$

For small angles $\sin(\theta) = \tan(\theta) = \theta$ holds, such that we can re-write F_y as

$$F_y = F\,\tan(\theta_2) - F\,\tan(\theta_1)$$

The tangent of the angle made by the string with the horizontal is the slope S of the curve formed by the string. We have

$$S = \tan(\theta) = \frac{\triangle y}{\triangle x} = \frac{\partial y}{\partial x}$$

where the r.h.s is the limit $\triangle x \to 0$, a partial derivative. This is the derivative of a function of several variables with respect to one of the variables, while the others are held constant. Then

$$F_y = F \triangle S = \mu \triangle x \frac{\partial^2 y}{\partial t^2}$$

where the r.h.s. is the mass $\mu \triangle x$ times the acceleration. Therefore,

$$F \frac{\triangle S}{\triangle x} = F \frac{\partial^2 y}{\partial x^2} = \mu \frac{\partial^2 y}{\partial t^2} \ .$$

This is the wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2} \quad \text{with} \quad v = \sqrt{\frac{F}{\mu}} \ .$$

By differentiation one can show that the wave equation is satsified by any function $y(x - v\,t)$.

Harmonic Waves

If one end of a string is attached to a vibrating fork that is moving up and down with an oscillation of frequency f, a sinusoidal wave propagates along the string (figure 15-7 of Tipler). It is called a harmonic wave. The distance after which the wave repeats itself, for example from crest to crest, is the wavelength λ.

As the wave propagates, each point moves up and down in simple harmonic motion, which is perpendicular to the direction of propagation. During one period $T = 1/f$, the wave moves a distance of one wavelength, so its speed is

$$v = \frac{\lambda}{T} = f\,\lambda\ .$$

Later we see that other waves are superpositions of harmonic waves. The wave function of a harmonic wave is

$$y(x,t) = A\,\sin\left[k\,(x - vt)\right] = A\,\sin(k\,x - \omega\,t)$$

where A is the amplitude, k the wave number, and for the angular frequency the following equations hold

$$\omega = k\,v = 2\pi\,f = \frac{2\pi}{T} = \frac{2\pi\,v}{\lambda}\ .$$

Energy of Harmonic Waves

The kinetic energy of a wave segment is

$$\triangle K = \frac{1}{2} \triangle m \, v_y^2 = \frac{1}{2} \mu \, \triangle x \left(\frac{\partial y}{\partial t} \right)^2 .$$

Using $y(x,t) = A \sin(k\,x - \omega\,t)$ we obtain $v_y = \partial y/\partial t = -\omega\,A \cos(k\,x - \omega\,t)$ and the kinetic energy is

$$\triangle K = \frac{1}{2} \mu\,\omega^2 A^2 \triangle x \cos^2(k\,x - \omega\,t) .$$

The potential energy is the work done in stretching the string, which for small oscillations can shown to be (Tipler, problem 123)

$$\triangle U = \frac{1}{2} F \triangle x \left(\frac{\partial y}{\partial x} \right)^2 \quad \text{where} \quad F \quad \text{is the tension.}$$

Using $\partial y/\partial x = k\,A \cos(k\,x - \omega\,t)$, and $F = \mu\,v^2 = \mu\,\omega^2/k^2$ the potential energy of the segment is

$$\begin{aligned} \triangle U &= \frac{1}{2} \left(\frac{\mu\,\omega^2}{k^2} \right) k^2 A^2 \triangle x \cos^2(k\,x - \omega\,t) \\ &= \frac{1}{2} \mu\,\omega^2 A^2 \triangle x \cos^2(k\,x - \omega\,t) \end{aligned}$$

Therefore, the total energy is

$$\triangle E = \triangle K + \triangle U = \mu\,\omega^2\,A^2\,\triangle x\,\cos^2(k\,x - \omega\,t)\ .$$

In contrast to simple harmonic motion, the energy is not constant, but moves. The average energy is

$$\triangle E_{\rm av} = \frac{1}{2}\,\mu\,\omega^2\,A^2\,\triangle x$$

which is the same results as for simple harmonic motion of a mass $\mu\,\triangle x$. The average rate at which energy is transmitted is the average power:

$$P_{\rm av} = \frac{dE_{\rm av}}{dt} = \frac{1}{2}\,\mu\,\omega^2\,A^2\,\frac{\triangle x}{\triangle t} = \frac{1}{2}\,\mu\,\omega^2\,A^2\,v$$

The average energy and power are proportional to the square of the amplitude.

Harmonic Sound Waves

Harmonic sound waves can be generated by a tuning fork or loudspeaker that is vibrating with simple harmonic motion. This causes displacements of molecules along the direction of motion, which lead to variations in the density and pressure. One can see (Tipler figure 15-10) that the pressure or density wave is 90^o out of phase with the displacement wave. Thus, the pressure is given by

$$p(x,t) = p_0 \sin(k\,x - \omega\,t - \pi/2) \ .$$

Waves in Three Dimensions

Wave may be generated by a point source moving up and down with harmonic motion. The wavelength is the distance between successive wave crests, which in this case are concentric circles, called wavefronts.

The motion of any set of wavefronts can be indicated by rays, which are directed perpendicular to the wave fronts (figure 15-12 of Tipler).

At a great distance from a point source, a small part of the wavefront can be approximated by a plane wave, for which the rays are parallel lines.

Standing Waves (Chapter 16-2)

When waves are confined in space, reflections at both ends cause the wave to travel in both directions. For a string or pipe, there are certain frequencies for which superposition results in a stationary pattern called standing wave. The frequencies that produce these patterns are called resoncance frequencies. Each such frequency with its accompanying wave function is called a mode of vibration. The lowest frequency produces the fundamental mode or first harmonic. For each frequency there are certain points on the string that do not move. Such points are called nodes. Midway between each pair of nodes is a point of maximum amplitude of vibration called an antinode.

String fixed at both ends (Figure 16-11 of Tipler):

The standing wave condition is

$$L = n\,\frac{\lambda_n}{2}, \quad n = 1, 2, 3, \ldots$$

and with $f_n\,\lambda_n = v$ the resonance frequencies become

$$f_n = n\,\frac{v}{2L} = n\,f_1, \quad n = 1, 2, 3, \ldots$$

where f_1 is the fundamental frequency. Example: pianos.