Superpositions of Waves (Chapter 16)

When two waves meet in space, they add algebraically (superposition). The superposition of harmonic waves is called interference. In 1801 Young observed the interference of light. Davisson and Germer observed in 1927 the interference of electron waves.

The principle of superposition:

When two or more waves combine, the resultant wave is the algebraic sum of the individual waves:

$$y_3(x,t) = y_1(x,t) + y_2(x,t)\ .$$

Examples: Figure 16-1 of Tipler.

Interference of Harmonic Waves:

Two wave sources that are in phase or have a constant phase difference are said to be coherent, otherwise they are said to be incoherent. We consider now two coherent waves

$$\begin{aligned} y_1 &= y_0 \sin(k\,x - \omega\,t) \\ y_2 &= y_0 \sin(k\,x - \omega\,t + \delta)\ . \end{aligned}$$

The resultant wave is

$$y_3 = y_1 + y_2 = y_0 \sin(k\,x - \omega\,t) + y_0 \sin(k\,x - \omega\,t + \delta)\ .$$

Using the trigonometric identity

$$\sin(\theta_1) + \sin(\theta_2) = 2\,\cos[(\theta_1 - \theta_2)/2]\,\sin[(\theta_1 + \theta_2)/2]$$

we get

$$y_3 = 2y_0\,\cos(\delta/2)\,\sin(k\,x - \omega\,t + \delta/2)\ .$$

The resulting wave has interesting properties:

If the two waves are in phase, $\delta = 0$, the amplitude of y_3 is $2y_0$, constructive interference (Figure 16-3 of Tipler).

If the two wave are 180^o out of phase, $\delta = \pi$, then $y_3 = 0$, destructive interference (Figure 16-4 of Tipler).

Beats:

This phenomenon is caused by the interference of sound waves with slightly different frequencies.What do we hear? For equal amplitudes we have at a fixed point, up to a phase constant, the pressure fluctuation

$$p = p_1 + p_2 = p_0 \sin(\omega_1 t) + p_0 \sin(\omega_2 t)$$

$$\begin{aligned} p &= 2p_0 \cos[(\omega_1 - \omega_2)\, t/2] \sin[(\omega_1 + \omega_2)\, t/2] \\ &= 2p_0 \cos[(\triangle\omega/2)\, t] \sin[(\omega_{\rm av}\, t] \end{aligned}$$

where $\triangle\omega = \omega_1 - \omega_2$ and $\omega_{\rm av} = (\omega_1 + \omega_2)/2$. The frequencies of the factors are

$$f_{\rm beat} = 2\triangle f = \frac{2\triangle\omega}{2\pi} \quad \text{and} \quad f_{\rm av} = \frac{2\omega_{\rm av}}{2\pi} \;.$$

The tone we hear has the average frequency $f_{\rm av}$, whose amplitude $2p_0\, cos(2\pi\, f_{\rm beat}\, t)$ is modulated by the beat frequency, which is much smaller than the average frequency (Figure 16-5 of Tipler). Beats can be used to tune a piano.

Phase Difference due to Path Difference:

The wave function from two coherent sources, oscillating in phase, can be written as (Figure 16-6 of Tipler)

$$p = p_1 + p_2 = p_0 \, \sin(k\, x_1 + \omega\, t) + p_0 \, \sin(k\, x_2 + \omega\, t) \ .$$

An example is given in Figure 16-8 of Tipler. The phase difference for these two wave function is

$$\delta = k\,(x_2 - x_1) = 2\pi \, \frac{\triangle x}{\lambda} \ .$$

The amplitude is $2p_0 \, \cos(\delta/2)$ and Figure 16-9 of Tipler shows how the intensity varies with the path difference.

The Double-Slit Experiment:

Interference of light is difficult to observe, because a light beam is usually the result of millions of atoms radiating incoherently. Coherence in optics is commonly achieved by splitting the light beam from a single source. One method of achieving this is by diffraction of a light beam by two slits in a barrier (Thomas Young 1801). The intensity of the resulting pattern can be calculated, see Figures 16-10 and 17-1 of Tipler for examples.

Wave Functions for Standing Waves

Standing wave occur due to the superposition of the reflected waves. When a sting vibrates in its nth mode, a point on the string moves with simple harmonic motion. Therefore, the wave function is given by

$$y(x, t) = A_n(x)\,\cos(\omega_n\, t + \delta_n)$$

where ω_n is the angular frequency, δ_n the phase constant, and $A(x)$ the amplitude, which depends on the location on the string. At an instant where the vibration is at its maximum amplitude, the shape of the string is

$$A_n(x) = A_n\,\sin(k_n\, x)$$

where $k_n = 2\pi/\lambda_n$ is the wave number. The wave function for a standing wave in the nth harmonic can thus be written

$$y_n(x, t) = A_n\,\sin(k_n\, x)\,\cos(\omega_n\, t + \delta_n)\ .$$

Superpositions of Standing Waves

In general, a vibrating system does not vibrate in a single harmonic mode. Instead, the motion consists of a mixture of the allowed harmonics and the wave function is a linear combination of the harmonic wave functions:

$$y(x,t) = \sum_n A_n \sin(k_n x) \cos(\omega_n t + \delta_n)$$

where $k_n = 2\pi/\lambda_n$, $\omega_n = 2\pi f_n$, and A_n, δ_n are constants which depend on the initial position and velocity of the string. Interestingly each wave, which fulfills the appropriate boundary conditions (here $y = 0$ at $x = 0$ and $x = L$), can be expanded in this way.

Harmonic Analysis and Synthesis:

Waves can be analyzed in terms of harmonics. Example: Figure 16-24 of Tipler shows the relative intensities for a tuning fork, a clarinet, and an oboe, each playing a tone at a fundamental frequency of 440 Hz.

The inverse is harmonics synthesis, the construction of a periodic wave from harmonic components. Example: Figure 16-25 and 16-26 of Tipler.

Wave Packets and Dispersion: Pulses, which are not periodic, can also be expanded into sinusoidal waves of different frequencies. However a continuous distributions of frequencies rather than a discrete set of harmonics is needed. These are wave packets. The characteristic feature of a wave pulse is that it has a beginning and an end. If the duration of the pulse is $\triangle t$, the range of frequencies $\triangle\omega$, needed to describe the impulse, is given by the relation

$$\triangle\omega\,\triangle t \sim 1\ .$$

E.g., if $\triangle t$ is very small, $\triangle\omega$ is very large and vice versa.

A wave pulse produced by a source of duration $\triangle t$ has a width $\triangle x = v\,\triangle t$ in space, where v is the wave speed. A range of frequencies $\triangle\omega$ implies a range of wave numbers $\triangle k = \triangle\omega / v$. Therefore, $\triangle\omega\,\triangle t \sim 1$ implies

$$\triangle k\,\triangle x \sim 1\ .$$

If a wave packet is to maintain its shape as it travels, all of the components must travel at the same speed. A medium where this happens is called non-dispersive medium.

Air is a non-dispersive medium for sound waves, but solids and liquids are generally not.

A familiar example for the dispersion of light waves is the rainbow.

When the speed of the wave component depends only slightly on the their wavelength, the wave packet changes shape only slowly as it travels. However, the speed of the wave packet, called group velocity, is not the same as the (average) speed of the components, called phase velocity. For example, the group velocity of surface waves in deep water is half the phase velocity.