Momentum Conservation

The Center of Mass (CM):

The CM $\vec{r}_{cm}$ moves as if all the external forces acting on the system were acting on the total mass $M$ of the system located at this point. In particular, it moves with constant velocity, if the external forces acting on the system add to zero.

Definition:

$$M \vec{r}_{cm} = \sum_{i=1}^{n} m_i \vec{r}_i \quad \text{where} \quad M = \sum_{i=1}^{n} m_i .$$

Here the sum is over the particles of the system, $m_i$ are the masses and $\vec{r}_i$ are the position vectors of the particles. In case of a continuous object, this becomes

$$M \vec{r}_{cm} = \int \vec{r} \, dm$$

where $dm$ is the position element of mass located at position $\vec{r}$.
Momentum:

The mass of a particle times its velocity is called momentum

\[ \vec{p} = m \vec{v}. \]

Newton's second law can be written as

\[ \vec{F}_{\text{net}} = \frac{d\vec{p}}{dt} = \frac{d(m\vec{v})}{dt} = m \frac{d\vec{v}}{dt} = m \vec{a} \]

as the masses of our particles have been constant.

The total momentum \( \vec{P} \) of a system is the sum of the momenta of the individual particles:

\[ \vec{P} = \sum_{i=1}^{n} \vec{p}_i = \sum_{i=1}^{n} m_i \vec{v}_i = M \vec{v}_{\text{cm}} \]

Differentiating this equation with respect to time, we obtain

\[ \frac{d\vec{P}}{dt} = M \frac{d\vec{v}_{\text{cm}}}{dt} = M \vec{a}_{\text{cm}} = \vec{F}_{\text{net,ext}} \]

The law of momentum conservation: When the net external force is zero, the total momentum is constant

\[ \vec{F}_{\text{net,ext}} = 0 \quad \Rightarrow \quad \vec{P} = \text{constant}. \]
Exercise:

Do problems 2 and 3 of CAPA set 10 again!
Rotation

1. The angular velocity $\vec{\omega}$. Direction: Right-hand-rule.

2. In accordance with the right-hand-rule the torque is defined as a vector: $\vec{\tau} = \vec{r} \times \vec{F}$.

3. Angular Momentum Definition: $\vec{L} = \vec{r} \times \vec{p}$.

Like the torque angular momentum is defined with respect to the point in space where the position vector $\vec{r}$ originates. For a rotation around a symmetry axis we find $\vec{L} = I \vec{\omega}$ (magnitude $L = I \omega$).

4. Rotational kinetic energy: $K_{\text{rot}} = \frac{1}{2} I \omega^2$.

Examples of moments of inertia:

$I_{\text{ss}} = (2/5) m R^2$ for a solid sphere.

$I_{\text{hs}} = (2/3) m R^2$ for hollow sphere.
Torque and Angular Momentum

The net external torque acting on a system equals the rate of change of the angular momentum of the system:

$$\sum_i \tau_{i, \text{ext}} = \frac{dL}{dt}$$

Conservation of Angular Momentum

If the net external torque acting on a system is zero, the total angular momentum of the system is constant.

$$\frac{d\vec{L}}{dt} = \vec{\tau}_{\text{net}} = 0 \quad \Rightarrow \quad \vec{L} = \text{constant}.$$
Gravity

Nothing on Gravity will be on the final!
Oscillations

Oscillations occur when a system is disturbed from stable equilibrium. Examples: Water waves, clock pendulum, string on musical instruments, sound waves, electric currents, ...

Simple Harmonic Motion

Example: Hooke’s law for a spring.

\[ F_x = m a = -k x = m \frac{d^2x}{dt^2} \]

\[ a = \frac{d^2x}{dt^2} = -\frac{k}{m} x \]

The acceleration is proportional to the displacement and is oppositely directed. This defines harmonic motion.

The time it takes to make a complete oscillation is called the period \( T \). The reciprocal of the period is the frequency

\[ f = \frac{1}{T} \]

The unit of frequency is the inverse second \( s^{-1} \), which is called a hertz \( Hz \).
Solution of the differential equation:

\[ x = x(t) = A \cos(\omega t + \delta) = A \sin(\omega t + \delta - \pi/2) \]

\( A, \omega \) and \( \delta \) are constants: \( A \) is the amplitude, \( \omega \) the angular frequency, and \( \delta \) the phase.

\[ v = v(t) = \frac{dx}{dt} = -\omega A \sin(\omega t + \delta) \]

\[ a = a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2} = -\omega^2 A \cos(\omega t + \delta) = -\omega^2 x \]

Therefore, for the spring

\[ \omega = \sqrt{\frac{k}{m}}. \]

Initial conditions: The amplitude \( A \) and the phase \( \delta \) are determined by the initial position \( x_0 \) and initial velocity \( v_0 \):

\[ x_0 = A \cos(\delta) \quad \text{and} \quad v_0 = -\omega A \sin(\delta). \]

In particular, for the initial position \( x_0 = x_{\text{max}} = A \), the maximum displacement, we have \( \delta = 0 \Rightarrow v_0 = 0 \).
The period $T$ is the time after which $x$ repeats:

$$x(t) = x(t + T) \Rightarrow \cos(\omega t + \delta) = \cos(\omega t + \omega T + \delta)$$

Therefore,

$$\omega T = 2\pi \Rightarrow \omega = \frac{2\pi}{T} = 2\pi f$$

is the relationship between the frequency and the angular frequency.