# Momentum Conservation

The Center of Mass (CM):

The CM $\vec{r}_{\rm cm}$ moves as if all the external foces acting on the system were acting on the total mass $M$ of the system located at this point. In particular, it moves with constant velocity, if the external forces acting on the system add to zero.

Definition:

$$M\,\vec{r}_{\rm cm} = \sum_{i=1}^{n} m_i\,\vec{r}_i \quad \text{where} \quad M = \sum_{i=1}^{n} m_i \ .$$

Here the sum is over the particles of the system, $m_i$ are the masses and $\vec{r}_i$ are the position vectors of the particles. In case of a continuous object, this becomes

$$M\,\vec{r}_{\rm cm} = \int \vec{r}\,dm$$

where $dm$ is the position element of mass located at position $\vec{r}$.

## Momentum:

The mass of a particle times its velocity is called momentum

$$\vec{p} = m\,\vec{v}\ .$$

Newton's second law can be written as

$$\vec{F}_{\rm net} = \frac{d\vec{p}}{dt} = \frac{d(m\vec{v})}{dt} = m\,\frac{d\vec{v}}{dt} = m\,\vec{a}$$

as the masses of our particles have been constant.

The total momentum $\vec{P}$ of a system is the sum of the momenta of the individual particles:

$$\vec{P} = \sum_{i=1}^{n} \vec{p}_i = \sum_{i=1}^{n} m_i\,\vec{v}_i = M\,\vec{v}_{\rm cm}$$

Differentiating this equation with respect to time, we obtain

$$\frac{d\vec{P}}{dt} = M\,\frac{d\vec{v}_{\rm cm}}{dt} = M\,\vec{a}_{\rm cm} = \vec{F}_{\rm net,ext}$$

The law of momentum conservation: When the net external force is zero, the total momentum is constant

$$\vec{F}_{\rm net,ext} = 0 \quad \Rightarrow \quad \vec{P} = \text{constant.}$$

## Exercise:

Do problems 2 and 3 of CAPA set 10 again!

# Rotation

1. The angular velocity $\vec{\omega}$. Direction: Right-hand-rule.

2. In accordance with the right-hand-rule the torque is defined as a vector: $\vec{\tau} = \vec{r} \times \vec{F}$ .

3. Angular Momentum Definition: $\vec{L} = \vec{r} \times \vec{p}$ .

   Like the torque angular momentum is defined with respect to the point in space where the position vector $\vec{r}$ originates. For a rotation around a symmetry axis we find $\vec{L} = I\,\vec{\omega}$ (magnitude $L = I\,\omega$).

4. Rotational kinetic energy: $K_{\rm rot} = \frac{1}{2} I\,\omega^2$ .

Examples of moments of inertia:

$I_{\rm ss} = (2/5)\,m\,R^2$ for a solid sphere.

$I_{\rm hs} = (2/3)\,m\,R^2$ for hollow sphere.

## Torque and Angular Momentum

The net external torque acting on a system equals the rate of change of the angular momentum of the system:

$$\sum_i \vec{\tau}_{i,\text{ext}} = \frac{d\vec{L}}{dt}$$

### Conservation of Angular Momentum

If the net external torque acting on a system is zero, the total angular momentum of the system is constant.

$$\frac{d\vec{L}}{dt} = \vec{\tau}_{\text{net}} = 0 \quad \Rightarrow \quad \vec{L} = \text{constant.}$$

# Gravity

Nothing on Gravity will be on the final!

# Oscillations

Oscillations occur when a system is disturbed from stable equilibrium. Examples: Water waves, clock pendulum, string on musical instruments, sound waves, electric currents, ...

## Simple Harmonic Motion

Example: Hooke's law for a spring.

$$F_x = m\,a = -k\,x = m\,\frac{d^2x}{dt^2}$$

$$a = \frac{d^2x}{dt^2} = -\frac{k}{m}\,x$$

The acceleration is proportional to the displacement and is oppositely directed. This defines harmonic motion.

The time it takes to make a complete oscillation is called the period $T$. The reciprocal of the period is the frequency

$$f = \frac{1}{T}$$

The unit of frequency is the inverse second $s^{-1}$, which is called a hertz $Hz$.

Solution of the differential equation:

$$x = x(t) = A\,\cos(\omega\,t + \delta) = A\,\sin(\omega\,t + \delta - \pi/2)$$

$A$, $\omega$ and $\delta$ are constants: $A$ is the amplitude, $\omega$ the angular frequencey, and $\delta$ the phase.

$$v = v(t) = \frac{dx}{dt} = -\omega\,A\,\sin(\omega\,t + \delta)$$

$$a = a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2} = -\omega^2\,A\,\cos(\omega\,t + \delta) = -\omega^2\,x$$

Therefore, for the spring

$$\omega = \sqrt{\frac{k}{m}}\ .$$

Initial conditions: The amplitude $A$ and the phase $\delta$ are determined by the initial position $x_0$ and initial velocity $v_0$:

$$x_0 = A\,\cos(\delta) \ \ \text{and} \ \ v_0 = -\omega\,A\,\sin(\delta)\ .$$

In particular, for the initial position $x_0 = x_{\max} = A$, the maximum displacement, we have $\delta = 0 \Rightarrow v_0 = 0$.

The period $T$ is the time after which $x$ repeats:

$$x(t) = x(t + T) \Rightarrow \cos(\omega\, t + \delta) = \cos(\omega\, t + \omega\, T + \delta)$$

Therefore,

$$\omega\, T = 2\pi \Rightarrow \omega = \frac{2\pi}{T} = 2\pi\, f$$

is the relationship between the frequency and the angular frequency.