Two-Dimensional Relativistic Kinematics

We consider the two inertial frames: K with coordinates $(t, \vec{x})$ and K' with coordinates $(t', \vec{x}^{\,\prime})$, and uniform relative motion $\vec{v}$. We demand that at time $t = t' = 0$ their two origins coincide. As we just have seen, this can be achieved. Imagine a spherical shell of radiation originating at time $t = 0$ from $\vec{x} = \vec{x}^{\,\prime} = 0$. The propagation of the wavefront is described by

$$c^2 t^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = 0 \quad \text{in } K, \tag{1}$$

and by

$$c^2 t'^2 - (x'^{\,1})^2 - (x'^{\,2})^2 - (x'^{\,3})'^2 = 0 \quad \text{in } K'. \tag{2}$$

It is customary to define $x^0 = ct$, $x'^{\,0} = ct'$ and $\beta = v/c$. For simplicity we chose now $\vec{v}$ in x-direction and restrict the discussion of the light wavefront to the x-axis. We are looking for a linear transformation

$$\begin{pmatrix} x'^{\,0} \\ x'^{\,1} \end{pmatrix} = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \end{pmatrix} \tag{3}$$

so that

$$(x'^{\,0})^2 - (x'^{\,1})^2 = (x^0)^2 - (x^1)^2 \tag{4}$$

holds for all x^0, x^1. Choosing $\begin{pmatrix} x^0 \\ x^1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ gives

$$(a_{00})^2 - (a_{10})^2 = 1 \;\Rightarrow\; a_{00} = \cosh(\zeta),\; a_{10} = \pm\sinh(\zeta) \tag{5}$$

and choosing $\begin{pmatrix} x^0 \\ x^1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ gives

$$(a_{01})^2 - (a_{11})^2 = -1 \;\Rightarrow\; a_{11} = \cosh(\eta),\; a_{01} = \pm\sinh(\eta)\ . \tag{6}$$

Using now $\begin{pmatrix} x^0 \\ x^1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ yields

$$[\cosh(\zeta) + \sinh(\eta)]^2 - [\sinh(\zeta) + \cosh(\eta)]^2 = 0 \;\Rightarrow\; \zeta = \eta.$$

In equation (5) $a_{10} = -\sinh(\zeta)$ is conventionally used. We end up with

$$\begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} = \begin{pmatrix} \cosh(\zeta) & -\sinh(\zeta) \\ -\sinh(\zeta) & \cosh(\zeta) \end{pmatrix}, \tag{7}$$

where ζ is called *rapidity* or *boost* variable, and has the interpretation of an angle in a hyperbolic geometry. For our present purposes no knowledge of hyperbolic geometries is required. The physical interpretation follows straightforward by remembering that, seen from K', the origin $x^1 = 0$ of K moves with constant velocity $-v$. In K' this corresponds to the equation ($x'^{\,0}$ and $x'^{\,1}$ denoting the origin of K)

$$-\beta = -\frac{v}{c} = \frac{x'^{\,1}}{x'^{\,0}} = -\frac{x^0 \sinh(\zeta)}{x^0 \cosh(\zeta)} = \tanh(\zeta).$$

Defining $\gamma = \cosh(\zeta)$ we obtain

$$\gamma\,\beta = \sinh(\zeta) \quad \text{and} \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}.$$

Hence, the transformation (3) follows in an often stated form

$$x^{'\,0} = \gamma\,(x^0 - \beta\,x^1)\,, \tag{8}$$

$$x^{'\,1} = \gamma\,(x^1 - \beta\,x^0)\,. \tag{9}$$

Equation (7), (8,9) are called *Lorentz transformations*. Lorentz discovered them first in his studies of electrodynamics, but it remained due to Einstein [1] to fully understand their physical meaning. We may perform two subsequent Lorentz transformations with rapidity ζ_1 and ζ_2. They combine as follows:

$$\begin{pmatrix} +\cosh(\zeta_2) & -\sinh(\zeta_2) \\ -\sinh(\zeta_2) & +\cosh(\zeta_2) \end{pmatrix} \begin{pmatrix} +\cosh(\zeta_1) & -\sinh(\zeta_1) \\ -\sinh(\zeta_1) & +\cosh(\zeta_1) \end{pmatrix}$$

$$= \begin{pmatrix} +\cosh(\zeta_2+\zeta_1) & -\sinh(\zeta_2+\zeta_1) \\ -\sinh(\zeta_2+\zeta_1) & +\cosh(\zeta_2+\zeta_1) \end{pmatrix}\ . \tag{10}$$

The rapidities add up in the same way as velocities due under Galilei transformations or angles due under rotations about the same axis. Note that the inverse to the transformation with rapidity ζ_1 is obtained for $\zeta_2 = -zeta_1$.

In coming section we shall need a slight generalization, allowing for the fact that at $x^0 = x^1 = 0$ does not have to coincide with $x^{'\,0} = x^{'\,1} = 0$. The light radiation may originate in K at (x^0_0, x^1_0) and in K' at $(x^{'\,0}_0, x^{'\,1}_0)$. This generalizes the wavefront equation (4) to

$$(x^{'\,0} - x^{'\,0}_0)^2 - (x^{'\,1} - x^{'\,1}_0)^2 = (x^0 - x^0_0)^2 - (x^1 - x^1_0)^2,$$

and the Lorentz transformations (8,9) become

$$(x^{'\,0} - x^{'\,0}_0) = \gamma\,[(x^0 - x^0_0) - \beta\,(x^1 - x^1_0)]\,, \tag{11}$$

$$(x^{'\,1} - x^{'\,1}_0) = \gamma\,[(x^1 - x^1_0) - \beta\,(x^0 - x^0_0)]\,. \tag{12}$$

Or, using the rapidity variable and matrix notation:

$$\begin{pmatrix} x^{'\,0} - x^{'\,0}_0 \\ x^{'\,1} - x^{'\,1}_0 \end{pmatrix} = \begin{pmatrix} \cosh(\zeta) & -\sinh(\zeta) \\ -\sinh(\zeta) & \cosh(\zeta) \end{pmatrix} \begin{pmatrix} x^0 - x^0_0 \\ x^1 - x^1_0 \end{pmatrix} \tag{13}$$

In this form they are called *Poincaré transformations*. Independently of Einstein, Poincaré had developed similar ideas, but pursued a much more cautious approach.

An immediate consequence of the Lorentz transformations is the *time dilatation*: A moving clock ticks slower. In K the position of the origin of K' is given by

$$x^1 = v\,x^0/c = \tanh(\zeta)\,x^0$$

and the Lorentz transformation (3,7) gives

$$x^{'\,0} = x^0/\cosh(\zeta)\ . \tag{14}$$

This works also the other way round. In K' the position of the origin of K is given by

$$x'^{\,1} = -\tanh(\zeta)\, x'^{\,0}$$

and with this relation between $x'^{\,1}$ and $x'^{\,0}$ the inverse Lorentz transformation gives

$$x^0 = x'^{\,0} / \cosh(\zeta)\ .$$

There is no paradox, because equal times at separate points in one frame are not equal in another (remember that the definition of time in one frame relies already on the constant speed of light). In particle physics the effect is day by day observed for the lifetimes of unstable particles. To test time dilatation for macrosciopic clocks, we have to send a clock on a roundtrip. For this an infinitesimal form of equation (14) is needed.

References

[1] A. Einstein, *Zur Elektrodynamik bewegter Körper*, Annalen der Physik 17 (1905) 891–921.