ADVANCED DYNAMICS — PHY 4241/5227

HOME AND CLASS WORK – SET 5

(January 30, 2009)

(20) Calculate

$\delta_x L$ for the Lagrangian of two harmonically bound particles in 1D

$$L = \frac{m_1}{2}(\dot{x}_1)^2 + \frac{m_2}{2}(\dot{x}_2)^2 - k\,(x_1 - x_2)^2 \ .$$

Is there a conservation law? Due Februar 2 in class (2 points).

(21) Calculate

$\delta_{x^i} L$ $(i = 1, \dots, 3)$ for the Lagrangian of the 3D Kepler problem

$$L = \frac{m_1}{2}(\dot{\vec{x}}_1)^2 + \frac{m_2}{2}(\dot{\vec{x}}_2)^2 - \frac{g\, m_1 m_2}{|\vec{x}_1 - \vec{x}_2|}$$

Are there any associated conservation laws? Due February 4 in class (4 points).

Read M&T chapter 7.10 (due February 4).

(22) Consider the spherical pendulum of mass m again and

(a) Use the Legendre transformation

$$H(\theta, \phi, p_\theta, p_\phi) = \dot{\theta} p_\theta + \dot{\phi} p_\phi - L(\theta, \phi, \dot{\theta}, \dot{\phi}) \ ,$$

to construct the Hamiltonian of the system and show that it is identical to the energy $E = T + V$.

(b) Write down Hamilton's equations of motions for the system and identify a conserved quantity.

Due February 9 before class (8 points).

(23) Liouville's Theorem: We define the velocity in phase space as a $2n$-dimensional vector $\vec{v} = (\dot{q}_1, \dots, \dot{q}_n, \dot{p}_1, \dots, \dot{p}_n)$. A large collection of particles can be described by their density in phase space $\rho(q_1, \dots, q_n, p_1, \dots, p_n)$. If their are no sources or sinks, we have a conserved current

$$\frac{\partial \rho}{\partial t} + \nabla\,(\vec{v}\,\rho) = 0 \qquad (1)$$

where $\nabla = \left(\frac{\partial}{\partial q_1}, \dots, \frac{\partial}{\partial q_n}, \frac{\partial}{\partial p_1}, \dots, \frac{\partial}{\partial p_n} \right)$ is the gradient in phase space.

(a) Expand (1) in sums of partial derivatives (you get five terms when you keep coordinates and momenta in separate contributions). (b) Use Hamilton's equations of motion to show that two terms cancel out. (c) Combine the remaining terms to

$$\frac{d\rho}{dt} = 0 \ \ \text{(Liouville's Theorem)}. \qquad (2)$$

Due February 6 in class (6 points). Read M&T chapter 7.12 (due February 9).