Prepared by: Piekarewicz
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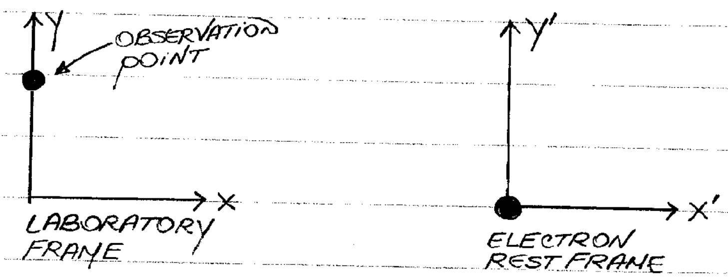
PHY-4241/5227 FINAL EXAM

PROBLEM 1.



(a) THE ELECTRIC AND MAGNETIC FIELDS MEASURED BY AN INERTIAL OBSERVER MOVING WITH THE ELECTRON ARE SIMPLE,

$$\vec{E}'(\vec{x}',t') = \frac{e}{r'^2}\hat{r}' = \frac{e\vec{r}'}{r'^3} = \frac{e}{(x'^2+y'^2+z'^2)^{3/2}}(x'\hat{x}+y'\hat{y}+z'\hat{z})$$

$$\vec{B}'(\vec{x}',t') \equiv 0$$

HENCE,

$$\boxed{\begin{aligned} E'_x &= e\frac{x'}{(x'^2+y'^2+z'^2)^{3/2}}\,; & B'_x &= 0 \\ E'_y &= e\frac{y'}{(x'^2+y'^2+z'^2)^{3/2}}\,; & B'_y &= 0 \\ E'_z &= e\frac{z'}{(x'^2+y'^2+z'^2)^{3/2}}\,; & B'_z &= 0 \end{aligned}}$$

(b) WE DERIVED IN CLASS THE FOLLOWING EXPRESSION FOR THE ELECTROMAGNETIC TENSOR

$$F'_{\mu\nu}(\vec{x}',t') = \begin{pmatrix} 0 & E'_x & E'_y & E'_z \\ -E'_x & 0 & -B'_z & B'_y \\ -E'_y & B'_z & 0 & -B'_x \\ -E'_z & -B'_y & B'_x & 0 \end{pmatrix} = \begin{pmatrix} 0 & E'_x & E'_y & E'_z \\ -E'_x & 0 & 0 & 0 \\ -E'_y & 0 & 0 & 0 \\ -E'_z & 0 & 0 & 0 \end{pmatrix}$$

WITH E'_x, E'_y, AND E'_z AS GIVEN IN PART (a).

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(c) TO START, WE INTRODUCE THE LORENTZ TRANSFORMATION FROM THE ELECTRON REST FRAME TO THE LABORATORY FRAME. THAT IS,

$$\Lambda^{\mu}_{\ \nu}(-\beta\hat{x}) = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad [\text{NOTE } x^{\mu} = \Lambda^{\mu}_{\ \nu} x'^{\nu}]$$

THEN,

$$F_{\mu\nu} = \Lambda^{\alpha}_{\ \mu} F'_{\alpha\beta} \Lambda^{\beta}_{\ \nu} = (\Lambda F' \Lambda)_{\mu\nu}$$

THE MAPLE OUTPUT (ENCLOSED AT THE END OF THE PROBLEM) REVEALS THAT THE ELECTRIC AND MAGNETIC FIELDS IN THE LABORATORY FRAME ARE GIVEN BY

$$E_x = E'_x \; ; \quad E_y = \gamma(E'_y - \beta B'_z) \; ; \quad E_z = \gamma(E'_z + \beta B'_y)$$
$$B_x = B'_x \; ; \quad B_y = \gamma(B'_y + \beta E'_z) \; ; \quad B_z = \gamma(B'_z - \beta E'_y)$$

WHICH IN OUR PARTICULAR CASE REDUCES TO THE FOLLOWING EXPRESSIONS:

$$E_x = E'_x \; ; \quad E_y = \gamma E'_y \; ; \quad E_z = \gamma E'_z$$
$$B_x = 0 \; ; \quad B_y = \beta\gamma E'_z \; ; \quad B_z = -\beta\gamma E'_y$$

WE ARE NOT DONE YET, SINCE WE NEED TO TRANSFORM THE SPACE-TIME COORDINATES FROM THE ELECTRON REST FRAME TO THE LABORATORY FRAME. FOR THE PARTICULAR CASE OF $x=0, y=y_0, z=0$ WE OBTAIN:

$$x' = \gamma(x - vt) = -\gamma v t$$
$$y' = y = y_0$$
$$z' = z = 0$$

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FINALLY, WE OBTAIN

$$E_x = -e\frac{\gamma v t}{[(\gamma v t)^2 + y_0^2]^{3/2}} \; ; \quad B_x = 0$$

$$E_y = +e\frac{\gamma y_0}{[(\gamma v t)^2 + y_0^2]^{3/2}} \; ; \quad B_y = 0$$

$$E_z = 0 \; ; \quad B_z = -e\frac{\beta \gamma y_0}{[\gamma v t + y_0^2]^{3/2}}$$

Partial solution to Problem 1

```
> restart:
> with(linalg):
> gama := 1/sqrt(1-beta^2):
> Lambda := matrix(4,4,[gama,beta*gama,0,0,
                        beta*gama,gama,0,0,
                        0,0,1,0,0,0,0,1]);
```

$$\Lambda := \begin{bmatrix} \frac{1}{\sqrt{1-\beta^2}} & \frac{\beta}{\sqrt{1-\beta^2}} & 0 & 0 \\ \frac{\beta}{\sqrt{1-\beta^2}} & \frac{1}{\sqrt{1-\beta^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

```
> Fmunu1 := matrix(4,4,[0,+Ex,+Ey,+Ez,-Ex,0,-Bz,+By,
                        -Ey,+Bz,0,-Bx,-Ez,-By,+Bx,0]);
```

$$Fmunu1 := \begin{bmatrix} 0 & Ex & Ey & Ez \\ -Ex & 0 & -Bz & By \\ -Ey & Bz & 0 & -Bx \\ -Ez & -By & Bx & 0 \end{bmatrix}$$

```
> Fmunu0 := map(simplify,evalm(Lambda &* Fmunu1 &* Lambda));
```

$$Fmunu0 = \begin{bmatrix} 0 & Ex & -\frac{-Ey+\beta\, Bz}{\sqrt{1-\beta^2}} & \frac{Ez+\beta\, By}{\sqrt{1-\beta^2}} \\ -Ex & 0 & \frac{\beta\, Ey-Bz}{\sqrt{1-\beta^2}} & \frac{\beta\, Ez+By}{\sqrt{1-\beta^2}} \\ \frac{-Ey+\beta\, Bz}{\sqrt{1-\beta^2}} & -\frac{\beta\, Ey-Bz}{\sqrt{1-\beta^2}} & 0 & -Bx \\ -\frac{Ez+\beta\, By}{\sqrt{1-\beta^2}} & -\frac{\beta\, Ez+By}{\sqrt{1-\beta^2}} & Bx & 0 \end{bmatrix}$$

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PROBLEM 2

(a)

$$L = \frac{1}{2}M\left(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2\right) - \frac{1}{2}k\left[x_1^2 + (x_2 - x_1)^2 + (x_3 - x_2)^2 + x_3^2\right]$$

$$= \frac{1}{2}M\left(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2\right) - \frac{1}{2}k\left[x_1^2 + (x_1^2 - 2x_1x_2 + x_2^2) + (x_2^2 - 2x_2x_3 + x_3^2) + x_3^2\right]$$

OR

$$L = \frac{1}{2}\dot{\eta}^T \hat{T} \dot{\eta} - \frac{1}{2}\eta^T \hat{V} \eta \;; \quad \eta^T \equiv (x_1, x_2, x_3)$$

WITH

$$\boxed{\hat{T} = M\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \equiv M\hat{t} \;; \quad \hat{V} = k\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \equiv k\hat{v}}$$

(b) THE EIGENVALUE EQUATION IS GIVEN BY:

$$0 = \left(-\omega^2 \hat{T} + \hat{V}\right)|\eta\rangle = \left(-\omega^2 M\hat{t} + k\hat{v}\right)|\eta\rangle$$

OR

$$\hat{v}|\eta\rangle = \lambda|\eta\rangle \;; \quad \lambda \equiv \omega^2/\omega_0^2 \quad (\omega_0^2 \equiv k/M)$$

FROM THIS EQUATION WE FIND NON-TRIVIAL SOLUTIONS BY DEMANDING THAT

$$\det(\hat{v} - \lambda\hat{1}) = \begin{vmatrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{vmatrix} = 0$$

OR

$$0 = (2-\lambda)\left[(2-\lambda)^2 - 1\right] - \left[(2-\lambda)\right] = (2-\lambda)\left[(2-\lambda)^2 - 2\right]$$

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OR $\lambda_1 = 2$ AND $\lambda_{2,3} = 2 \pm \sqrt{2}$. NOW TO THE EIGENVECTORS,

a) $\lambda_1 = 2$

$$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = 0 \Rightarrow \eta_2 = 0 \text{ AND } \eta_1 + \eta_3 = 0$$

So,

$$|\eta_1\rangle = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

b) $\lambda_2 = 2 + \sqrt{2}$

$$\begin{pmatrix} -\sqrt{2} & -1 & 0 \\ -1 & -\sqrt{2} & -1 \\ 0 & -1 & -\sqrt{2} \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = 0 \Rightarrow \begin{array}{l} \sqrt{2}\eta_1 + \eta_2 = 0 \rightarrow \eta_2 = -\sqrt{2}\eta_1 \\ \eta_2 + \sqrt{2}\eta_3 = 0 \rightarrow \eta_3 = -\frac{\eta_2}{\sqrt{2}} = \eta_1 \end{array}$$

So,

$$|\eta_2\rangle = \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

c) $\lambda_3 = 2 - \sqrt{2}$

$$\begin{pmatrix} \sqrt{2} & -1 & 0 \\ -1 & \sqrt{2} & -1 \\ 0 & -1 & \sqrt{2} \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = 0 \Rightarrow \begin{array}{l} \sqrt{2}\eta_1 - \eta_2 = 0 \rightarrow \eta_2 = \sqrt{2}\eta_1 \\ -\eta_2 + \sqrt{2}\eta_3 = 0 \rightarrow \eta_3 = \frac{\eta_2}{\sqrt{2}}\ \eta_1 \end{array}$$

So,

$$|\eta_3\rangle = \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$$

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OR

$$\boxed{\begin{aligned} \omega_1^2 &= 2\omega_0^2 ; \quad |\omega_1\rangle = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \\ \omega_2^2 &= (2+\sqrt{2})\,\omega_0^2 ; \quad |\omega_2\rangle = \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} \\ \omega_3^2 &= (2-\sqrt{2})\,\omega_0^2 ; \quad |\omega_3\rangle = \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} \end{aligned}}$$

NOTE THAT (AS REQUIRED):

- ALL EIGENVECTORS ARE ORTHOGONAL
- THE TRACE (WHICH EQUALS 6) REMAINS INVARIANT
- THE DETERMINANT (WHICH EQUALS 4) REMAINS INVARIANT

(c) THE MOST GENERAL SOLUTION IS GIVEN BY:

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = (A_1 \cos\omega_1 t + B_1 \sin\omega_1 t) \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$+ (A_2 \cos\omega_2 t + B_2 \sin\omega_2 t) \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

$$+ (A_3 \cos\omega_3 t + B_3 \sin\omega_3 t) \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$$

NOTE THAT SINCE $\dot{x}_1(0) = \dot{x}_2(0) = \dot{x}_3(0) = 0$, THEN $B_1 = B_2 = B_3 = 0$.

TO DETERMINE THE As WE USE:

$$\left.\begin{aligned} 1 &= A_1 + A_2 + A_3 \\ 0 &= -\sqrt{2}A_2 + \sqrt{2}A_3 \\ 0 &= -A_1 + A_2 + A_3 \end{aligned}\right\} \Rightarrow A_1 = \frac{1}{2} ; \quad A_2 = A_3 = \frac{1}{4}$$

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FINALLY, WE OBTAIN

$$\boxed{\begin{aligned} X_1(t) &= \frac{1}{2}\cos\omega_1 t + \frac{1}{4}\cos\omega_2 t + \frac{1}{4}\cos\omega_3 t \\ X_2(t) &= -\frac{1}{2\sqrt{2}}\cos\omega_2 t + \frac{1}{2\sqrt{2}}\cos\omega_3 t \\ X_3(t) &= -\frac{1}{2}\cos\omega_1 t + \frac{1}{4}\cos\omega_2 t + \frac{1}{4}\cos\omega_3 t \end{aligned}}$$