

PROBLEM 23

The Matrix L is defined by:

$$L = \begin{pmatrix} l^0_0 & l^0_1 & l^0_2 & l^0_3 \\ l^1_0 & l^1_1 & l^1_2 & l^1_3 \\ l^2_0 & l^2_1 & l^2_2 & l^2_3 \\ l^3_0 & l^3_1 & l^3_2 & l^3_3 \end{pmatrix}.$$

(a)–Calculate $-gL$. (b)–Write down the transpose matrix $\tilde{L}$. (c)–Calculate $\tilde{L}g$.
 (d)–Compare (a) and (c) to find the general form of L (*i.e.* use $\tilde{L}g = -gL$).

(a)–The matrix g is defined by:

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

–By matrix multiplication:

$$-gL = \begin{pmatrix} -l^0_0 & -l^0_1 & -l^0_2 & -l^0_3 \\ l^1_0 & l^1_1 & l^1_2 & l^1_3 \\ l^2_0 & l^2_1 & l^2_2 & l^2_3 \\ l^3_0 & l^3_1 & l^3_2 & l^3_3 \end{pmatrix}.$$

(b)–The transpose of L is:

$$\tilde{L} = \begin{pmatrix} l^0_0 & l^1_0 & l^2_0 & l^3_0 \\ l^0_1 & l^1_1 & l^2_1 & l^3_1 \\ l^0_2 & l^1_2 & l^2_2 & l^3_2 \\ l^0_3 & l^1_3 & l^2_3 & l^3_3 \end{pmatrix}.$$

(c)–By matrix multiplication:

$$\tilde{L}g = \begin{pmatrix} l^0_0 & -l^1_0 & -l^2_0 & -l^3_0 \\ l^0_1 & -l^1_1 & -l^2_1 & -l^3_1 \\ l^0_2 & -l^1_2 & -l^2_2 & -l^3_2 \\ l^0_3 & -l^1_3 & -l^2_3 & -l^3_3 \end{pmatrix}.$$

(d)–Set $-gL = \tilde{L}g$ and compare the components:

$$\begin{pmatrix} -l^0_0 = l^0_0 & -l^0_1 = -l^1_0 & -l^0_2 = -l^2_0 & -l^0_3 = -l^3_0 \\ l^1_0 = l^0_1 & l^1_1 = -l^1_1 & l^1_2 = -l^2_1 & l^1_3 = -l^3_1 \\ l^2_0 = l^0_2 & l^2_1 = -l^1_2 & l^2_2 = -l^2_2 & l^2_3 = -l^3_2 \\ l^3_0 = l^0_3 & l^3_1 = -l^1_3 & l^3_2 = -l^2_3 & l^3_3 = -l^3_3 \end{pmatrix}. \quad (1)$$

–The resulting matrix is:

$$\begin{pmatrix} 0 & l^0_1 & l^0_2 & l^0_3 \\ l^0_1 & 0 & l^1_2 & l^1_3 \\ l^0_2 & -l^1_2 & 0 & l^2_3 \\ l^0_3 & -l^1_3 & -l^2_3 & 0 \end{pmatrix}. \quad (2)$$

(e)–Obtain the same result by discussing the elements of $g^{\alpha\beta} \tilde{l}_\beta^\gamma g_{\gamma\delta} = -l^\alpha_\delta$. Performing the contractions we get $\tilde{l}^\alpha_\delta = -l^\alpha_\delta$ and these equations are identical with the equations (1), namely (no summations, $a = 0, 1, 2, 3$, $i = 1, 2, 3$ and $j = 1, 2, 3$):

$$\tilde{l}^a_a = l^a_a = +l^a_a \Rightarrow l^a_a = -l^a_a = 0 ,$$

$$\tilde{l}^i_0 = l^i_0 = -l^0_i \Rightarrow l^0_i = +l^i_0 ,$$

$$\tilde{l}^i_j = l^i_j = +l^j_i \Rightarrow l^j_i = -l^i_j$$

and the resulting matrix is again (2).