ADVANCED DYNAMICS — PHY–4241/5227

HOMEWORK 5

(February 3, 2003)
Due on Monday, February 10, 2003

PROBLEM 13

By using the following definition for the antisymmetric **Levi-Civita** symbol:

$$\varepsilon_{ijk} = \begin{cases} +1, & \text{if } ijk \text{ is an even permutation of 123;} \\ -1, & \text{if } ijk \text{ is an odd permutation of 123;} \\ 0, & \text{otherwise;} \end{cases}$$

show, in a line or two, that

a) $\nabla \times \Big(\nabla\Phi(\mathbf{x})\Big) = 0$.

b) $\nabla \cdot \Big(\nabla \times \mathbf{A}(\mathbf{x})\Big) = 0$.

c) $\nabla \times \Big(\nabla \times \mathbf{A}(\mathbf{x})\Big) = \nabla\Big(\nabla \cdot \mathbf{A}(\mathbf{x})\Big) - \nabla^2\mathbf{A}(\mathbf{x})$.

You may find useful the following two important identities:

$$\begin{aligned} \varepsilon_{ijk}\varepsilon_{ijl} &= 2\delta_{kl} \, , \\ \varepsilon_{ijk}\varepsilon_{ilm} &= \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl} \, . \end{aligned}$$

PROBLEM 14

The concept of a **gauge transformation** emerges from the observation that clearly distinct vector potentials can generate the same exact magnetic field.

a) Show that if two vector potentials, $\mathbf{A}_1(\mathbf{x})$ and $\mathbf{A}_2(\mathbf{x})$, differ by the gradient of a scalar function $\Lambda(\mathbf{x})$, then both potentials generate the same exact magnetic field $\mathbf{B}(\mathbf{x})$. That is, if $\mathbf{A}_2(\mathbf{x}) = \mathbf{A}_1(\mathbf{x}) + \nabla\Lambda(\mathbf{x})$ and $\mathbf{B}(\mathbf{x}) = \nabla \times \mathbf{A}_1(\mathbf{x})$, then $\mathbf{B}(\mathbf{x}) = \nabla \times \mathbf{A}_2(\mathbf{x})$.

Now consider the following three vector potentials with B a constant:

$$\begin{aligned} \mathbf{A}_1(\mathbf{x}) &= +B\Big(x\hat{\mathbf{y}} - y\hat{\mathbf{x}}\Big)/2 \, , \\ \mathbf{A}_2(\mathbf{x}) &= +Bx\hat{\mathbf{y}} \, , \\ \mathbf{A}_3(\mathbf{x}) &= -By\hat{\mathbf{x}} \, , \end{aligned}$$

b) Show that all three vector potentials satisfy: $\nabla \times \mathbf{A} = B\hat{\mathbf{z}}$ and $\nabla \cdot \mathbf{A} = 0$.

c) Find the three gauge transformations, *i.e.,* the three scalar functions $\Lambda_{12}(\mathbf{x})$, $\Lambda_{23}(\mathbf{x})$, and $\Lambda_{31}(\mathbf{x})$, that connect the above three gauge-equivalent vector potentials.

PROBLEM 15

Show that the refraction of light as it passes from one medium (with ε_1 and μ_1) to another one (with ε_2 and μ_2) obeys Fermat's principle of least time.