Special Relativity and Maxwell Equations

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Chapter 1

A self-contained summary of the theory of special relativity is given, which provides the space-time frame for classical electrodynamics. Historically [2] special relativity emerged out of electromagnetism. Nowadays, is deserves to be emphasized that special relativity restricts severely the possibilities for electromagnetic equations.

1.1 Special Relativity

Let us deal with space and time in vacuum. The conventional time unit is one

$$\textit{second}\ [s]. \tag{1.1}$$

Here, and in the following abbreviations for units are placed in brackets [ ]. For a long time period the second was defined in terms of the rotation of the earth as $\frac{1}{60} \times \frac{1}{60} \times \frac{1}{24}$ of the mean solar day. Nowadays most accurate time measurements rely on atomic clocks. They work by tuning a electric frequency into resonance with some atomic transition. Consequently, the second has been re–defined, so that the frequency of the light between the two hyperfine levels of the ground state of the cesium ^{132}Cs atom is now exactly 9,192,631,770 cycles per second.

Special relativity is founded on two basic postulates:

1. Galilee invariance: The laws of nature are independent of any uniform, translational motion of the reference frame.

This postulate gives rise to a triple infinite set of reference frames moving with constant velocities relative to one another. They are called *inertial frames*. For a freely moving body, i.e. a body which is not acted upon by an external force, inertial systems exist. The differential equations which describe physical laws take the same form in all inertial frames. *Galilee invariance* was known long before Einstein.

2. The speed c of light in empty space is independent of the motion of its source.

The second Postulate was introduced by Einstein 1905 [2]. It implies that c takes the same constant value in all inertial frames. Transformations between inertial frames follow, which have far reaching physical consequences.

The distance unit

$$1\ meter\ [m] = 100\ centimeters\ [cm] \tag{1.2}$$

was originally defined by two scratches on a bar made of platinum–iridium alloy kept at the International Bureau of Weights and Measures in Sèvres, France. As measurements of the speed of light have become increasingly accurate, it has become most appropriate to exploit Postulate 2 to define the distance unit. The standard meter is now defined [6] as the distance traveled by light in empty space during the time of 1/299,792,458 [s]. This makes the speed of light exactly

$$c = 299,792,458\ [m/s]. \tag{1.3}$$

1.1.1 Natural Units

The units for *second* (1.1 and *meter* (1.2) are not independent, as the speed of light is an universal constant. This allows to define natural units, which are frequently used in nuclear, particle and astro physics. They define

$$c = 1 \tag{1.4}$$

as a dimensionless constant, and

$$1\ [s] = 299,792,458\ [m]$$

holds. The advantage of natural units is that factors of c disappear in calculations. The disadvantage is that, for converting back to conventional units, the appropriate factors have to be recovered by dimensional analysis. For instance, if time is given in seconds $x = t$ in natural units converts to $x = ct$ with x in meters and c given by (1.3).

1.1.2 Definition of distances and synchronization of clocks

Let us relate the introduced concepts of Galilee invariance and of a constant speed of light to physical measurements and reduce measurements of spatial distances to time measurements. Let us consider an inertial frame K with coordinates $(t, \vec{x})$. We like to place observers at rest at different places $\vec{x}$ in K. The observers are equipped with clocks of identical making, to define the time t at $\vec{x}$. The origin $\vec{x} = 0$, in other notation $(t, \vec{0})$, of K is defined by placing observer O_0 and his clock there. We like to place another observer O_1 at $\vec{x}_1$ to define $(t, \vec{x}_1)$. How can O_1 know to be at $\vec{x}_1$? By using a mirror he can reflect light flashed by observer O_0 at him. Observer O_0 may measure the polar and azimuthal angles (θ, ϕ) at which he emits the light and

$$|\vec{x}_1| = c \triangle t/2\,,$$

where $\triangle t$ is the time light needs to travel to O_1 and back. This determines $\vec{x}_1$ and he can signal this information to O_1. By repeating the measurement, he can make sure that O_1 is not moving with respect to K. For an idealized, force free environment the observers will then never start moving with respect to one another. O_1 synchronizes his clock by setting it to

$$t_1 = t_0 + |\vec{x}_1|/c$$

at the instant receiving the signal. If O_0 flashes again his instant time t_{01} over to O_1, the clock of O_1 will show time $t_{11} = t_{01} + |\vec{x}_1|/c$ at the instant of receiving the signal. In the same way the time t can be defined at any desired point $\vec{x}$ in K.

Now we consider an inertial frame K' with coordinates $(t', \vec{x}')$, moving with constant velocity $\vec{v}$ with respect to K. The origin of K' is defined through a third observer O'_0. What does it mean that O'_0 moves with constant velocity $\vec{v}$ with respect to O_0? At times t^e_{01} and t^e_{02} observer O_0 may flash light signals (the superscript e stands for "emit") at O'_0, which are reflected and arrive back after time intervals $\triangle t_{01}$ and $\triangle t_{02}$. From principle 2 it follows that the reflected light needs the same time (measured by O_0 in K) to travel from O'_0 to O_0, as it needed to travel from O_0 to O'_0. Hence, O_0 concludes that O'_0 received the signal at

$$t_{0i} = t^e_{0i} + \triangle t_{0i}/2,\ \ (i = 1, 2) \tag{1.5}$$

in the O_0 time. This simple equation becomes quite complicated for non-relativistic physics, because the speed on the return path would then be distinct from that on the arrival path (consider for instance elastic scattering of a very light particle on a heavy surface). The constant velocity of light implies that relativistic distance measurements are simpler than such non-relativistic measurements. For observer O_0 the positions $\vec{x}_{01}$ and $\vec{x}_{02}$ are now defined through the angles (θ_{0i}, ϕ_{0i}) and

$$|\vec{x}_{0i}| = \triangle t_{0i}\, c/2,\ \ (i = 1, 2)\,. \tag{1.6}$$

For the assumed force free environment observer O_0 can conclude that O'_0 moves with respect to him with uniform velocity

$$\vec{v} = (\vec{x}_{02} - \vec{x}_{01})/(t_{02} - t_{01})\,. \tag{1.7}$$

Actually, one measurement is sufficient to obtain the velocity when one employs the relativistic Doppler effect as discussed later in section 1.1.8. O_0 may repeat the procedure at later times to check that O'_o moves indeed with uniform velocity.

Similarly, observer O'_0 finds out that O_0 moves with velocity $\vec{v}\,' = -\vec{v}$. According to principle 1, observers in K' can now go ahead to define t' for any point $\vec{x}\,'$ in K'. The equation of motion for the origin of K' is as observed by O_0 is

$$\vec{x}\,(\vec{x}\,' = 0) = \vec{x}_0 + \vec{v}\,t, \tag{1.8}$$

with $\vec{x}_0 = \vec{x}_{01} - \vec{v}\, t_{01}$, expressesing the fact that for $t = t_{01}$ observer O_0' is at $\vec{x}_{01}$. Shifting his space convention by a constant vector, observer O_0 can achieve $\vec{x}_0 = 0$, so that equation (1.8) becomes

$$\vec{x}\,(\vec{x}^{\,'} = 0) = \vec{v}\, t.$$

Similarly, observer O_0' may choose his space convention so that

$$\vec{x}^{\,'}\,(\vec{x} = 0) = -\vec{v}\, t'$$

holds.

1.1.3 Lorentz invariance and Minkowski space

Having defined time and space operationally, let us focus on a more abstract discussion. We consider the two inertial frames with uniform relative motion $\vec{v}$: K with coordinates $(t, \vec{x})$ and K' with coordinates $(t', \vec{x}\,')$. We demand that at time $t = t' = 0$ their two origins coincide. Now, imagine a spherical shell of radiation originating at time $t = 0$ from $\vec{x} = \vec{x}\,' = 0$. The propagation of the wavefront is described by

$$c^2 t^2 - x^2 - y^2 - z^2 = 0 \quad \text{in } K, \tag{1.9}$$

and by

$$c^2 t'^2 - x'^2 - y'^2 - z'^2 = 0 \quad \text{in } K'. \tag{1.10}$$

We define 4-vectors $(\alpha = 0, 1, 2, 3)$ by

$$(x^\alpha) = \begin{pmatrix} ct \\ \vec{x} \end{pmatrix} \quad \text{and} \quad (x_\alpha) = (ct,\ -\vec{x})\ . \tag{1.11}$$

Due to a more general notation, which is explained in section 1.1.5, the components x^α are called *contravariant* and the components x_α *covariant*. In matrix notation the contravariant 4-vector (x^α) is represented by a column and the covariant 4-vector (x_α) as a row.

The *Einstein summation convention* is defined by

$$x_\alpha x^\alpha = \sum_{\alpha=0}^{3} x_\alpha x^\alpha, \tag{1.12}$$

and will be employed from here on. Equations (1.9) and (1.10) read then

$$x_\alpha x^\alpha = x'_\alpha x'^\alpha = 0\ . \tag{1.13}$$

(Homogeneous) Lorentz transformations are defined as the group of transformations which leave the distance $s^2 = x_\alpha x^\alpha$ invariant:

$$x_\alpha x^\alpha = x'_\alpha x'^\alpha = s^2\ . \tag{1.14}$$

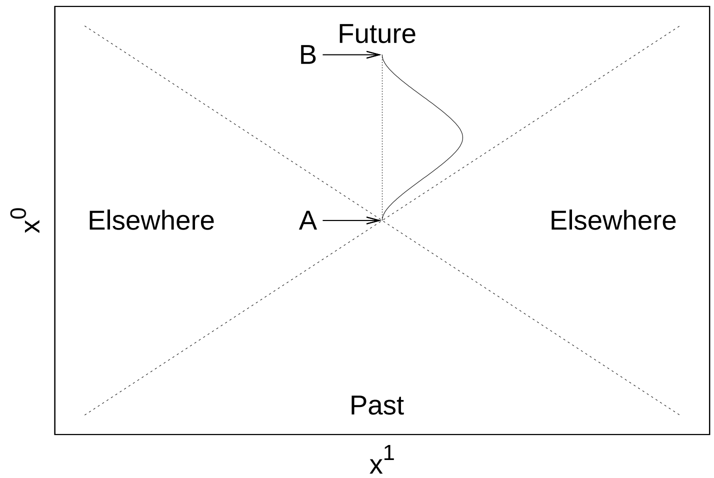

Figure 1.1: Minkowski space: Seen from the spacetime point A at the origin, the spacetime points in the forward light cone are in the future, those in the backward light cone are in the past and the spacelike points are "elsewhere", because their time-ordering depends on the inertial frame chosen. Paths of two clocks which separate at the origin (the straight line one stays at rest) and merge again at a future space-time point B are also indicated. For the paths shown the clock moved along the curved path will, at B, show an elapsed time of about 70% of the elapsed time shown by the other clock (*i.e.* the one which stays at rest).

This equation implies (1.13, but the reverse is not true. An additional transformation, which leave (1.13, but not (1.14), invariant is $x'^{\alpha} = \lambda\, x^{\alpha}$, which is interpreted as a scale transformation.

If the initial condition $t' = 0$ and $\vec{x}\,'(\vec{x} = 0) = \vec{x}(\vec{x}\,') = 0$ for $t = 0$ is replaced by an arbitrary one, the equation $(x_\alpha - y_\alpha)(x^\alpha - y^\alpha) = (x'_\alpha - y'_\alpha)(x'^\alpha - y'^\alpha)$ still holds. *Inhomogeneous Lorentz* or *Poincaré* transformations are defined as the group of transformations which leave

$$s^2 = (x_\alpha - y_\alpha)(x^\alpha - y^\alpha) \quad \text{invariant.} \tag{1.15}$$

In contrast to the Lorentz transformations the Poincaré transformations include invariance under *translations*

$$x^\alpha \to x^\alpha + a^\alpha \;\; \text{and} \;\; y^\alpha \to y^\alpha + a^\alpha \tag{1.16}$$

where a^α is a constant vector. Independently of Einstein, Poincaré had developed similar ideas, but pursued a more cautious approach.

A fruitful concept is that of a 4-dimensional space-time, called *Minkowski space*. Equation (1.15) gives the invariant metric of this space. Compared to the norm of 4-dimensional

Euclidean space, the crucial difference is the relative minus sign between time and space components. The *light cone* of a 4-vector x_0^α is defined as the set of vectors x^α which satisfy

$$(x - x_0)^2 = (x_\alpha - x_{0\alpha})\,(x^\alpha - x_0^\alpha) = 0.$$

The light cone separates events which are *timelike* and *spacelike* with respect to x_0^α, namely

$$(x - x_0)^2 > 0 \ \text{ for timelike}$$

and

$$(x - x_0)^2 < 0 \ \text{ for spacelike.}$$

We shall see soon, compare equation (1.22), that the time ordering of spacelike points is distinct in different inertial frames, whereas it is the same for timelike points. For the choice $x_0^\alpha = 0$ this Minkowski space situation is depicted in figure 1.1. On the abscissa we have the projection of the three dimensional Euclidean space on $r = |\vec{x}|$. The regions *future* and *past* of this figure are the timelike points of $x_0 = 0$, whereas *elsewhere* are the spacelike points.

To understand special relativity in some depth, we have to explore Lorentz and Poincaré transformations in some details. Before we come to this, we consider the two-dimensional case and introduce some relevant calculus in the next two sections.

1.1.4 Two-Dimensional Relativistic Kinematics

We chose now $\vec{v}$ in x-direction and restrict the discussion of to the x-axis:

$$c^2t^2 - (x^1)^2 = c^2t'^2 - (x'^1)^2 \tag{1.17}$$

It is customary to define $x^0 = ct$, $x'^0 = ct'$ and $\beta = v/c$. We are looking for a linear transformation

$$\begin{pmatrix} x'^0 \\ x'^1 \end{pmatrix} = \begin{pmatrix} a & b \\ d & e \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \end{pmatrix} \tag{1.18}$$

which fulfills (1.17) for all x^0, x^1. Choosing $\begin{pmatrix} x^0 \\ x^1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ gives

$$a^2 - d^2 = 1 \ \Rightarrow \ a = \cosh(\zeta),\ d = \pm\sinh(\zeta) \tag{1.19}$$

and choosing $\begin{pmatrix} x^0 \\ x^1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ gives

$$b^2 - e^2 = -1 \ \Rightarrow \ e = \cosh(\eta),\ b = \pm\sinh(\eta)\ . \tag{1.20}$$

Using now $\begin{pmatrix} x^0 \\ x^1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ yields

$$[\cosh(\zeta) + \sinh(\eta)]^2 - [\sinh(\zeta) + \cosh(\eta)]^2 = 0 \ \Rightarrow \ \zeta = \eta.$$

In equation (1.19) $d = -\sinh(\zeta)$ is conventionally used. We end up with

$$\begin{pmatrix} a & b \\ d & e \end{pmatrix} = \begin{pmatrix} \cosh(\zeta) & -\sinh(\zeta) \\ -\sinh(\zeta) & \cosh(\zeta) \end{pmatrix}, \tag{1.21}$$

where ζ is called *rapidity* or *boost* variable, and has the interpretation of an angle in a hyperbolic geometry. For our present purposes no knowledge of hyperbolic geometries is required. In components (1.21) reads

$$x'^0 = +\cosh(\zeta)\, x^0 - \sinh(\zeta)\, x^1 \tag{1.22}$$

$$x'^1 = -\sinh(\zeta)\, x^0 + \cosh(\zeta)\, x^1 \tag{1.23}$$

An interesting feature of equation (1.71) is that for spacelike points, say $x^1 > x^0 > 0$, a value ζ_0 for the rapidity exists, so that

$$0 = +\cosh(\zeta)\, x^0 - \sinh(\zeta)\, x^1$$

and, therefore,

$$\mathrm{sign}(x'^0) = -\mathrm{sign}(x^0)$$

for $\zeta > \zeta_0$, *i.e.* the time-ordering becomes reversed, whereas for timelike points such a reversal of the time-ordering is impossible as then $|x^0 > | > |x^1|$. In figure 1.1 this is emphasized by calling the spacelike (with respect to $x_0 = 0$) region *elsewhere* in contrast to *future* and *past*.

The physical interpretation is straightforward. Seen from K, the origin $x'^1 = 0$ of K moves with constant velocity v. In K this corresponds to the equation

$$0 = -\sinh(\zeta)\, x^0 + \cosh(\zeta)\, x^1$$

and the rapidity is related to the velocity between the frames by

$$\beta = \frac{v}{c} = \frac{x^1}{x^0} = \frac{\sinh(\zeta)}{\cosh(\zeta)} = \tanh(\zeta)\ . \tag{1.24}$$

Another often used notation is

$$\gamma = \cosh(\zeta) = \frac{1}{\sqrt{1-\beta^2}} \quad \text{and} \quad \gamma\,\beta = \sinh(\zeta)\ . \tag{1.25}$$

Hence, the transformation (1.18) follows in the often stated form

$$x'^0 = \gamma\,(x^0 - \beta\, x^1)\,, \tag{1.26}$$

$$x'^1 = \gamma\,(x^1 - \beta\, x^0)\,. \tag{1.27}$$

These equation are called *Lorentz transformations*. Lorentz discovered them first in his studies of electrodynamics, but it remained due to Einstein [2] to fully understand their physical meaning. We may perform two subsequent Lorentz transformations with rapidity ζ_1 and ζ_2. They combine as follows:

$$\begin{pmatrix} +\cosh(\zeta_2) & -\sinh(\zeta_2) \\ -\sinh(\zeta_2) & +\cosh(\zeta_2) \end{pmatrix} \begin{pmatrix} +\cosh(\zeta_1) & -\sinh(\zeta_1) \\ -\sinh(\zeta_1) & +\cosh(\zeta_1) \end{pmatrix}$$

$$\begin{pmatrix} +\cosh(\zeta_2+\zeta_1) & -\sinh(\zeta_2+\zeta_1) \\ -\sinh(\zeta_2+\zeta_1) & +\cosh(\zeta_2+\zeta_1) \end{pmatrix} . \tag{1.28}$$

The rapidities add up as

$$\zeta = \zeta_1 + \zeta_2 \tag{1.29}$$

in the same way as velocities do under Galilei transformations or angles for rotations about the same axis. Note that the inverse to the transformation with rapidity ζ_1 is obtained for $\zeta_2 = -\zeta_1$. The relativistic addition of velocities follows from (1.29). Let $\beta_1 = \tanh(\zeta_1)$ and $\beta_2 = \tanh(\zeta_2)$, then

$$\beta = \tanh(\zeta_1 + \zeta_2) = \frac{\beta_1 + \beta_2}{1 + \beta_1 \beta_2} \tag{1.30}$$

holds. Another immediate consequence of the Lorentz transformations is the *time dilatation*: A moving clock ticks slower. In K the position of the origin of K' is given by

$$x^1 = v\, x^0/c = \tanh(\zeta)\, x^0$$

and the Lorentz transformation (1.22) gives

$$x'^{\,0} = \cosh(\zeta)\, x^0 - \sinh(\zeta)\, \tanh(\zeta)\, x^0 = \frac{\cosh^2(\zeta) - \sinh^2(\zeta)}{\cosh(\zeta)}\, x^0 = \frac{x^0}{\cosh(\zeta)} < x^0 . \tag{1.31}$$

This works also the other way round. In K' the position of the origin of K is given by

$$x'^{\,1} = -\tanh(\zeta)\, x'^{\,0}$$

and with this relation between $x'^{\,1}$ and $x'^{\,0}$ the inverse Lorentz transformation gives

$$x^0 = x'^{\,0}/\cosh(\zeta) .$$

There is no paradox, because equal times at separate points in one frame are not equal in another (remember that the definition of time in one frame relies already on the constant speed of light). In particle physics the effect is day by day observed for the lifetimes of unstable particles. To test time dilatation for macrosciopic clocks, we have to send a clock on a roundtrip. For this an infinitesimal form of equation (1.31) is needed.

Allowing for the fact that at $x^0 = x^1 = 0$ does not have to coincide with $x'^{\,0} = x'^{\,1} = 0$, we have Poincaré transformations. The light radiation may originate in K at (x^0_0, x^1_0) and in K' at $(x'^{\,0}_0, x'^{\,1}_0)$. This generalizes equation (1.17) to

$$(x'^{\,0} - x'^{\,0}_0)^2 - (x'^{\,1} - x'^{\,1}_0)^2 = (x^0 - x^0_0)^2 - (x^1 - x^1_0)^2,$$

and the Lorentz transformations become

$$(x'^0 - x_0'^0) = \gamma\,[(x^0 - x_0^0) - \beta\,(x^1 - x_0^1)]\,, \tag{1.32}$$

$$(x'^1 - x_0'^1) = \gamma\,[(x^1 - x_0^1) - \beta\,(x^0 - x_0^0)]\,. \tag{1.33}$$

Using the rapidity variable and matrix notation:

$$\begin{pmatrix} x'^0 - x_0'^0 \\ x'^1 - x_0'^1 \end{pmatrix} = \begin{pmatrix} \cosh(\zeta) & -\sinh(\zeta) \\ -\sinh(\zeta) & \cosh(\zeta) \end{pmatrix} \begin{pmatrix} x^0 - x_0^0 \\ x^1 - x_0^1 \end{pmatrix} \tag{1.34}$$

In addition we have invariance under translations (1.16).

Let us explore Minkowski space in more details. It allows to depict *world lines* of particles. A useful concept for a particle (or observer) traveling along its world line is its *proper time* or *eigenzeit*. Assume the particle moves with velocity $v(t)$, then $dx^1 = \beta\, dx^0$ holds, and the infinitesimal invariant along its 2D world line is

$$(ds)^2 = (dx^0)^2 - (dx^1)^2 = (c\,dt)^2\,(1-\beta^2)\ . \tag{1.35}$$

Each instantaneous rest frame of the particle is an inertial frame. The increment of time $d\tau$ in such an instantaneous rest frame is a Lorentz invariant quantity which takes the form

$$d\tau = dt\,\sqrt{1-\beta^2} = dt\,\gamma^{-1} = dt/\cosh\zeta\ , \tag{1.36}$$

where τ is called proper time. Clocks click by their proper time. As $\gamma(\tau) \geq 1$ time dilatation follows

$$t_2 - t_1 = \int_{\tau_1}^{\tau_2} \gamma(\tau)\,d\tau = \int_{\tau_1}^{\tau_2} \cosh\zeta(\tau)\,d\tau \geq \tau_2 - \tau_1. \tag{1.37}$$

A moving clock runs more slowly than a stationary clock. Now equation (1.37) applies to general paths of a clock, including those with acceleration. Relevant is that the entire derivation was done with respect to the inertial system in which the time coordinates t_2 and t_1 are defined. Two experimental examples for time dilatation are: (i) Time of flight of unstable particles in high energy scattering experiments, where these particles move at velocities close to the speed of light. (ii) Explicit verification through travel with atomic clocks on air planes [4, 9].

Next, let us discuss the relation between velocity and *acceleration*. Assume an acce- laration a in the instanteneous rest frame. To have convenient units we define $\alpha = a/c$ and

$$d\beta = d\zeta = \alpha\,d\tau \tag{1.38}$$

holds in the instantenous rest frame. The change in another frame follow from the addition theorem of velocities (1.30)

$$d\beta = \frac{\alpha d\tau + \beta}{1 + \alpha d\tau\,\beta} - \beta = \alpha\,(1-\beta^2)\,d\tau \tag{1.39}$$

and simpler for the rapidity

$$d\zeta = (\alpha\, d\tau + \zeta) - \zeta = \alpha\, d\tau \ . \tag{1.40}$$

Using the proper time, the change of the rapidity is analogue to the change of the velocity in non-relativistic mechanics,

$$\zeta - \zeta_0 = \int_{\tau_0}^{\tau} \alpha(\tau)\, d\tau \ . \tag{1.41}$$

In particular, if α is constant we can integrate and find

$$\zeta(\tau) = \alpha\tau + \zeta_0 \quad \text{for} \quad \alpha \quad \text{constant} \ . \tag{1.42}$$

1.1.5 Vector and tensor notation

One defines a general transformation $x \to x'$ through

$$x'^{\,\alpha} = x'^{\,\alpha}(x) = x'^{\,\alpha}\left(x^0, x^1, x^2, x^3\right), \ \alpha = 0, 1, 2, 3. \tag{1.43}$$

This means, $x'^{\,\alpha}$ is a function of four variables and, when it is needed, this function is assumed to be sufficiently often differentiable with respect to each of its arguments. In the following we consider the transformation properties of various quantities (scalars, vectors and tensors) under $x \to x'$.

A *scalar* is a single quantity whose value is not changed under the transformation (1.43). The proper time is and example.

A *4-vector* A^α, $(\alpha = 0, 1, 2, 3)$ is said *contravariant* if its components transform according to

$$A'^\alpha = \frac{\partial x'^\alpha}{\partial x^\beta} A^\beta. \tag{1.44}$$

An example is $A^\alpha = dx^\alpha$, where (1.44) reduces to the well–known rule for the differential of a function of several variable $(f^\alpha(x) = x'^\alpha(x))$:

$$dx'^\alpha = \frac{\partial x'^{\,\alpha}}{\partial x^\beta} dx^\beta.$$

Remark: In this general framework the vector x^α is not always contravariant. When a linear transformation

$$x'^{\,\alpha} = a^\alpha{}_\beta\, x^\beta$$

holds, i.e., with space-time independent coefficients $a^\alpha{}_\beta$, this is the case and one finds

$$\frac{\partial x'^{\,\alpha}}{\partial x^\beta} = a^\alpha{}_\beta \ .$$

In the present framework we are only interested in linear transformation. Space-time dependent transformations lead into general relativity.

A 4-vector is said *covariant* when it transforms like

$$B'_\alpha = \frac{\partial x^\beta}{\partial x'^\alpha} B_\beta. \tag{1.45}$$

An example is

$$B_\alpha = \partial_\alpha = \frac{\partial}{\partial x^\alpha}, \tag{1.46}$$

because of

$$\frac{\partial}{\partial x'^\alpha} = \frac{\partial x^\beta}{\partial x'^\alpha} \frac{\partial}{\partial x^\beta}.$$

The *inner* or *scalar product* of two vectors is defined as the product of the components of a covariant and a contravariant vector:

$$B \cdot A = B_\alpha A^\alpha. \tag{1.47}$$

It follows from (1.44) and (1.45) that the scalar product is an invariant under the transformation (1.43):

$$B' \cdot A' = \frac{\partial x^\beta}{\partial x'^\alpha} \frac{\partial x'^\alpha}{\partial x^\gamma} B_\beta A^\gamma = \frac{\partial x^\beta}{\partial x^\gamma} B_\beta A^\gamma = \delta^\beta{}_\gamma B_\beta A^\gamma = B \cdot A.$$

Here the *Kronecker delta* is defined by:

$$\delta^\alpha{}_\beta = \delta_\alpha{}^\beta = \begin{cases} 1 & \text{for} \quad \alpha = \beta, \\ 0 & \text{for} \quad \alpha \neq \beta. \end{cases} \tag{1.48}$$

Vectors are *rank* one tensors. *Tensors* of general rank k are quantities with k indices, like for instance

$$T^{\alpha_1 \alpha_2 \ldots}{}_{\ldots \alpha_i \ldots \alpha_k} \;.$$

The convention is that the upper indices transform contravariant and the lower transform covariant. For instance, a contravariant tensor of rank two $F^{\alpha\beta}$ consists of 16 quantities that transform according to

$$F'^{\alpha\beta} = \frac{\partial x'^\alpha}{\partial x^\gamma} \frac{\partial x'^\beta}{\partial x^\delta} F^{\gamma\delta}.$$

A covariant tensor of rank two $G_{\alpha\beta}$ transforms as

$$G'_{\alpha\beta} = \frac{\partial x^\gamma}{\partial x'^\alpha} \frac{\partial x^\delta}{\partial x'^\beta} G_{\gamma\delta}.$$

The *inner product* or *contraction* with respect to a pair of indices, either on the same tensor or between different tensors, is defined in analogy with (1.47). One index has to be contravariant and the other covariant.

A tensor $S^{\ldots\alpha\ldots\beta\ldots}$ is said to be symmetric in α and β when

$$S^{\ldots\alpha\ldots\beta\ldots} = S^{\ldots\beta\ldots\alpha\ldots}.$$

A tensor $A^{\dots\alpha\dots\beta\dots}$ is said to be antisymmetric in α and β when

$$A^{\dots\alpha\dots\beta\dots} = -A^{\dots\beta\dots\alpha\dots}.$$

Let $S^{\dots\alpha\dots\beta}$ be a symmetric and $A^{\dots\alpha\dots\beta}$ be an antisymmetric tensor. It holds

$$S^{\dots\alpha\dots\beta\dots}A_{\dots\alpha\dots\beta\dots} = 0. \tag{1.49}$$

Proof:

$$S^{\dots\alpha\dots\beta\dots}A_{\dots\alpha\dots\beta\dots} = -S^{\dots\beta\dots\alpha\dots}A_{\dots\beta\dots\alpha\dots} = -S^{\dots\alpha\dots\beta\dots}A_{\dots\alpha\dots\beta\dots},$$

and consequently zero. The first step exploits symmetry and antisymmetry, and the second step renames the summation indices. Every tensor can be written as a sum of its symmetric and antisymmetric parts in two if its indices

$$T^{\dots\alpha\dots\beta\dots} = T_S^{\dots\alpha\dots\beta\dots} + T_A^{\dots\alpha\dots\beta\dots} \tag{1.50}$$

by simply defining

$$T_S^{\dots\alpha\dots\beta\dots} = \frac{1}{2}\left(T^{\dots\alpha\dots\beta\dots} + T^{\dots\beta\dots\alpha\dots}\right) \text{ and } T_A^{\dots\alpha\dots\beta\dots} = \frac{1}{2}\left(T^{\dots\alpha\dots\beta\dots} - T^{\dots\beta\dots\alpha\dots}\right). \tag{1.51}$$

So far the results and definitions are general. We now specialize to Poincaré transformations. The specific geometry of the space–time of special relativity is defined by the invariant distance s^2, see equation (1.15). In differential form, the infinitesimal interval ds defines the proper time $c\,d\tau = ds$,

$$(ds)^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2. \tag{1.52}$$

Here we have used superscripts on the coordinates in accordance to our insight that dx^α is a contravariant vector. Introducing a *metric tensor* $g_{\alpha\beta}$ we re–write equation (1.52) as

$$(ds)^2 = g_{\alpha\beta}\,dx^\alpha dx^\beta. \tag{1.53}$$

Comparing (1.52) and (1.53) we see that for special relativity $g_{\alpha\beta}$ is diagonal:

$$g_{00} = 1,\ g_{11} = g_{22} = g_{33} = -1 \text{ and } g_{\alpha\beta} = 0 \text{ for } \alpha \neq \beta. \tag{1.54}$$

Comparing (1.53) with the invariant scalar product (1.47), we conclude that

$$x_\alpha = g_{\alpha\beta}\,x^\beta.$$

The covariant metric tensor lowers the indices, i.e. transforms a contravariant into a covariant vector. Correspondingly the contravariant metric tensor $g^{\alpha\beta}$ is defined to raise indices:

$$x^\alpha = g^{\alpha\beta}\,x_\beta.$$

The last two equations and the symmetry of $g_{\alpha\beta}$ imply

$$g_{\alpha\gamma}\, g^{\gamma\beta} = \delta_\alpha^{\ \beta}$$

for the contraction of contravariant and covariant metric tensors. This is solved by $g^{\alpha\beta}$ being the normalized co–factor of $g_{\alpha\beta}$. For the diagonal matrix (1.54) the result is simply

$$g^{\alpha\beta} = g_{\alpha\beta}. \tag{1.55}$$

Consequently the equations

$$A^\alpha = \begin{pmatrix} A^0 \\ \vec{A} \end{pmatrix}, \quad A_\alpha = (A^0, -\vec{A})$$

and, compare (1.46),

$$(\partial_\alpha) = \left(\frac{\partial}{c\partial t}, \nabla\right), \quad (\partial^\alpha) = \begin{pmatrix} \frac{\partial}{c\partial t} \\ \\ -\nabla \end{pmatrix}. \tag{1.56}$$

hold. It follows that the 4-divergence of a 4-vector

$$\partial^\alpha A_\alpha = \partial_\alpha A^\alpha = \frac{\partial A^0}{\partial x^0} + \nabla \cdot \vec{A}$$

and the d'Alembert (4-dimensional Laplace) operator

$$\Box = \partial_\alpha \partial^\alpha = \left(\frac{\partial}{\partial x^0}\right)^2 - \nabla^2$$

are invariants. Sometimes the notation $\triangle = \nabla^2$ is used for the (3-dimensional) Laplace operator.

1.1.6 Lorentz transformations

Let us now construct the Lorentz group. We seek a group of linear transformations

$$x'^\alpha = a^\alpha_{\ \beta}\, x^\beta, \quad (\Rightarrow \frac{\partial x'^\alpha}{\partial x^\beta} = a^\alpha_{\ \beta}) \tag{1.57}$$

such that the scalar product stays invariant:

$$x'_\alpha\, x'^\alpha = a_\alpha^{\ \beta} x_\beta\, a^\alpha_{\ \gamma} x^\gamma = x_\alpha\, x^\alpha = \delta^\beta_{\ \gamma} x_\beta x^\gamma.$$

As the $x_\beta x^\gamma$ are independent, this yields

$$a_\alpha^{\ \beta}\, a^\alpha_{\ \gamma} = \delta^\beta_{\ \gamma} \;\Leftrightarrow\; a_{\alpha\beta}\, a^\alpha_{\ \gamma} = g_{\beta\gamma} \;\Leftrightarrow\; a^\delta_{\ \beta}\, g_{\delta\alpha}\, a^\alpha_{\ \gamma} = g_{\beta\gamma}\ .$$

In matrix notation

$$\tilde{A} g A = g, \tag{1.58}$$

where $g = (g_{\beta\alpha})$ is given by (1.54),

$$A = (a^{\beta}{}_{\alpha}) = \begin{pmatrix} a^0{}_0 & a^0{}_1 & a^0{}_2 & a^0{}_3 \\ a^1{}_0 & a^1{}_1 & a^1{}_2 & a^1{}_3 \\ a^2{}_0 & a^2{}_1 & a^2{}_2 & a^2{}_3 \\ a^3{}_0 & a^3{}_1 & a^3{}_2 & a^3{}_3 \end{pmatrix} , \tag{1.59}$$

and $\tilde{A} = (\tilde{a}_{\beta}{}^{\alpha})$ with $\tilde{a}_{\beta}{}^{\alpha} = a^{\alpha}{}_{\beta}$ is the transpose of the matrix $A = (a^{\beta}{}_{\alpha})$, explicitly

$$\tilde{A} = (\tilde{a}_{\beta}{}^{\alpha}) = \begin{pmatrix} \tilde{a}_0{}^0 & \tilde{a}_0{}^1 & \tilde{a}_0{}^2 & \tilde{a}_0{}^3 \\ \tilde{a}_1{}^0 & \tilde{a}_1{}^1 & \tilde{a}_1{}^2 & \tilde{a}_1{}^3 \\ \tilde{a}_2{}^0 & \tilde{a}_2{}^1 & \tilde{a}_2{}^2 & \tilde{a}_2{}^3 \\ \tilde{a}_3{}^0 & \tilde{a}_3{}^1 & \tilde{a}_3{}^2 & \tilde{a}_3{}^3 \end{pmatrix} = \begin{pmatrix} a^0{}_0 & a^1{}_0 & a^2{}_0 & a^3{}_0 \\ a^0{}_1 & a^1{}_1 & a^2{}_1 & a^3{}_1 \\ a^0{}_2 & a^1{}_2 & a^2{}_2 & a^3{}_2 \\ a^0{}_3 & a^1{}_3 & a^2{}_3 & a^3{}_3 \end{pmatrix} . \tag{1.60}$$

For this definition of the transpose matrix the row indices are contravariant and the column indices are covariant, vice verse to the definition (1.11) for vectors and, similarly, ordinary matrices. Certain properties of the transformation matrix A can be deduced from (1.58). Taking the determinant of both sides gives us $\det(\tilde{A}gA) = \det(g)\det(A)^2 = \det(g)$. Since $\det(g) = -1$, we obtain

$$\det(A) = \pm 1. \tag{1.61}$$

One distinguishes two classes of transformations. *Proper* Lorentz transformations are continuously connected with the identity transformation $A = \mathbf{1}$. All other Lorentz transformations are *improper*. Proper transformations have necessarily $\det(A) = 1$. For improper Lorentz transformations it is sufficient, but not necessary, to have $\det(A) = -1$. For instance $A = -\mathbf{1}$ (space and time inversion) is an improper Lorentz transformation with $\det(A) = +1$.

Next the number of parameters, needed to specify completely a transformation in the group, follows from (1.58). Since A and g are 4×4 matrices, we have 16 equations for $4^2 = 16$ elements of A. But they are not all independent because of symmetry under transposition. The off-diagonal equations are identical in pairs. Therefore, we have $4+6 = 10$ linearly independent equations for the 16 elements of A. This means that there are *six free parameters*. In other words, the Lorentz group is a six–parameter group.

In the 19th century *Lie* invented the subsequent procedure to handle these parameters. Let us now consider only proper Lorentz transformations. To construct A explicitly, Lie makes the ansatz

$$A = e^L = \sum_{n=0}^{\infty} \frac{L^n}{n!},$$

where L is a 4×4 matrix. The determinant of A is

$$\det(A) = \det(e^L) = e^{\mathrm{Tr}(L)}. \tag{1.62}$$

Note that $\det(A) = +1$ implies that L is traceless. Equation (1.58) can be written

$$g\tilde{A}g = A^{-1}. \tag{1.63}$$

From the definition of L, $\tilde{L}$ and the fact that $g^2 = \mathbf{1}$ we have (note $(g\tilde{L}g)^n = g\tilde{L}^n g$ and $\mathbf{1} = \left(\sum_{n=0}^{\infty} L^n/n!\right)\left(\sum_{n=0}^{\infty}(-L)^n/n!\right)$)

$$\tilde{A} = e^{\tilde{L}}, \quad g\tilde{A}g = e^{g\tilde{L}g} \quad \text{and} \quad A^{-1} = e^{-L} \; .$$

Therefore, (1.63) is equivalent to

$$g\tilde{L}g = -L \quad \text{or} \quad \widetilde{(gL)} = -gL.$$

The matrix gL is thus antisymmetric and it is left as an exercise to show that the general form of L is:

$$L = \begin{pmatrix} 0 & l^0{}_1 & l^0{}_2 & l^0{}_3 \\ l^0{}_1 & 0 & l^1{}_2 & l^1{}_3 \\ l^0{}_2 & -l^1{}_2 & 0 & l^2{}_3 \\ l^0{}_3 & -l^1{}_3 & -l^2{}_3 & 0 \end{pmatrix} . \tag{1.64}$$

It is customary to expand L in terms of six *generators*:

$$L = -\sum_{i=1}^{3}(\omega_i S_i + \zeta_i K_i) \quad \text{and} \quad A = e^{-\sum_{i=1}^{3}(\omega_i S_i + \zeta_i K_i)}. \tag{1.65}$$

The matrices are defined by

$$S_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{1.66}$$

and

$$K_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \tag{1.67}$$

They satisfy the following *Lie algebra* commutation relations:

$$[S_i, S_j] = \sum_{k=1}^{3} \epsilon^{ijk} S_k, \quad [S_i, K_j] = \sum_{k=1}^{3} \epsilon^{ijk} K_k, \quad [K_i, K_j] = -\sum_{k=1}^{3} \epsilon^{ijk} S_k,$$

where the commutator of two matrices is defined by $[A, B] = AB - BA$. Further ϵ_{ijk} is the completely antisymmetric Levi–Cevita tensor. Its definition in n–dimensions is

$$\epsilon^{i_1 i_2 \ldots i_n} = \begin{cases} +1 \text{ for } (i_1, i_2, ..., i_n) \text{ being an even permutation of } (1, 2, ..., n), \\ -1 \text{ for } (i_1, i_2, ..., i_n) \text{ being an odd permutation of } (1, 2, ..., n), \\ 0 \text{ otherwise.} \end{cases} \tag{1.68}$$

To get the physical interpretation of equation (1.65) for A, it is suitable to work out simple examples. First, let $\vec{\zeta} = \omega_1 = \omega_2 = 0$ and $\omega_3 = \omega$. Then (this is left as exercise)

$$A = e^{-\omega S_3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega & \sin\omega & 0 \\ 0 & -\sin\omega & \cos\omega & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{1.69}$$

which describes a rotation by the *angle* ω (in the clockwise sense) around the $\hat{e}_3$ axis. Next, let $\vec{\omega} = \zeta_2 = \zeta_3 = 0$ and $\zeta_1 = \zeta$. Then

$$A = e^{-\zeta K_1} = \begin{pmatrix} \cosh\zeta & -\sinh\zeta & 0 & 0 \\ -\sinh\zeta & \cosh\zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.70}$$

is obtained, where ζ is known as the *boost parameter* or *rapidity*. The structure is reminiscent to a rotation, but with hyperbolic functions instead of circular, basically because of the relative negative sign between the space and time terms in eqn.(1.52). “Rotations” in the $x^0 - x^i$ planes are boosts and governed by an hyperbolic gemometry, whereas rotations in the $x^i - x^j$ $(i \neq j)$ planes are governed by the ordinary Euclidean geometry.

Finally, note that the parameters ω_i, ζ_i, $(i = 1, 2, 3)$ turn out to be real, as equation (1.57) implies that the elements of A have to be real. In the next subsection relativistic kinematics is discussed in more details.

1.1.7 Basic relativistic kinematics

The matrix (1.70) gives the Lorentz boost transformation, which we discussed before in the 2D context (1.22),

$$x'^0 = x^0 \cosh(\zeta) - x^1 \sinh(\zeta), \tag{1.71}$$

$$x'^1 = -x^0 \sinh(\zeta) + x^1 \cosh(\zeta), \tag{1.72}$$

$$x'^i = x^i, \;\; (i = 2, 3). \tag{1.73}$$

To find the transformation law of an arbitrary vector $\vec{A}$ in case of a general relative velocity $\vec{v}$, it is convenient to decompose $\vec{A}$ into components parallel and perpendicular to $\vec{\beta} = \vec{v}/c$. Let $\hat{\beta}$ be the unit vector in $\vec{\beta}$ direction, then

$$\vec{A} = A^{\|}\hat{\beta} + \vec{A}^{\perp} \;\; \text{with} \;\; A^{\|} = \hat{\beta}\vec{A}.$$

Then the Lorentz transformation law is simply

$$A'^0 = A^0 \cosh(\zeta) - A^{\|} \sinh(\zeta) = \gamma(\;\; A^0 - \beta A^{\|}), \tag{1.74}$$

$$A'^{\|} = -A^0 \sinh(\zeta) + A^{\|} \cosh(\zeta) = \gamma(-\beta A^0 + A^{\|}), \tag{1.75}$$

$$\vec{A}'^{\perp} = \vec{A}^{\perp}. \tag{1.76}$$

Here I have reserved the subscript notation $A_{\|}$ and $\vec{A}_{\perp}$ for use in connection with covariant vectors: $A_{\|} = -A^{\|}$ and $\vec{A}_{\perp} = -\vec{A}^{\perp}$. We proceed deriving the addition theorem of velocities. Assume a particle moves with respect to K' with velocity $\vec{u}\,'$:

$$x'^i = c^{-1} u'^i x'^0.$$

Equations (1.26), (1.27) imply

$$\gamma\,(x^1-\beta x^0)=c^{-1}u'^1\gamma\,(x^0-\beta x^1).$$

Sorting with respect to x^1 and x^0 gives

$$\gamma\,\left(1+\frac{u'^1v}{c^2}\right)\,x^1=c^{-1}\gamma\,(u'^1+v)\,x^0\,.$$

Using the definition of the velocity in K, $\vec{x}=c^{-1}\vec{u}\,x^0$, gives

$$u^1=c\,\frac{x^1}{x^0}=\frac{u'^1+v}{1+\frac{u'^1v}{c^2}}\,. \tag{1.77}$$

Along similar lines, we obtain for the two other components

$$u^i=\frac{u'^i}{\gamma\left(1+\frac{u'^1v}{c^2}\right)},\quad (i=2,3). \tag{1.78}$$

To derive these equations, $\vec{v}$ was chosen along to the x^1-axis. For general $\vec{v}$ one only has to decompose $\vec{u}$ into its components parallel and perpendicular to the $\vec{v}$

$$\vec{u}=u^{\|}\hat{v}+\vec{u}^{\perp}\,,$$

where $\hat{v}$ is the unit vector in $\vec{v}$ direction, and obtains

$$u^{\|}=\frac{u'^{\|}+v}{1+\frac{u'^{\|}v}{c^2}}\quad\text{and}\quad\vec{u}^{\perp}=\frac{\vec{u}'^{\perp}}{\gamma\,\left(1+\frac{u'^{\|}v}{c^2}\right)}. \tag{1.79}$$

From this addition theorem of velocities it is obvious that the velocity itself is not part of of a 4-vector. The relativistic generalization is given in subsection (1.1.9). It is left as an exercise to relate these equations to the addition theorem for the rapidity (1.29.

The concepts of *world lines* in Minkowski space and *proper time* (*eigenzeit* generalized immediately to 4D. Assume the particle moves with velocity $\vec{v}(t)$, then $d\vec{x}=\vec{\beta}dx^0$ holds, and the infinitesimal invariant along its world line is

$$(ds)^2=(dx^0)^2-(d\vec{x})^2=(c\,dt)^2\,(1-\beta^2) \tag{1.80}$$

and the relations (1.36) and (1.37) hold as in 2D.

1.1.8 Plane waves and the relativistic Doppler effect

Let us choose coordinates with respect to an inertial frame K. In complex notation a plane wave is defined by the equation

$$W(x)=W(x^0,\vec{x})=W_0\,\exp[\,i\,(k^0\,x^0-\vec{k}\,\vec{x})\,]\,, \tag{1.81}$$

where $W_0 = U_0 + i\,V_0$ is a complex *amplitude*. The vector $\vec{k}$ is called *wave vector*. It becomes a 4-vector (k^α) by identifying

$$k^0 = \omega/c \tag{1.82}$$

as its zero-component, where ω is the *angular frequency* of the wave. Waves of the form (1.81) may either propagate in a medium (water, air, shock waves, etc.) or in vacuum (light waves, particle waves in quantum mechanics). We are interested in the latter case, as the other defines a preferred inertial frame, namely the one where the medium is at rest. The *phase* of the wave is defined by

$$\Phi(x) = \Phi(x^0, \vec{x}) = k^0\,x^0 - \vec{k}\,\vec{x} = \omega\,t - \vec{k}\,\vec{x}\,. \tag{1.83}$$

When (k^α) is a 4-vector, it follows that the phase is a scalar, *invariant* under Lorentz transformations

$$\Phi'(x') = k'_\alpha\,x'^{\,\alpha} = k_\alpha\,x^\alpha = \Phi(x)\,. \tag{1.84}$$

That this is correct can be seen as follows: For an observer at a fixed position $\vec{x}$ (note the term $\vec{k}\,\vec{x}$ is then constant) the wave performs a periodic motion with *period*

$$T = \frac{2\pi}{\omega} = \frac{1}{\nu}\,, \tag{1.85}$$

where ν is the *frequency*. In particular, the phase (and hence the wave) takes identical values on the two-dimensional hyperplanes perpendicular to $\vec{k}$. Namely, let $\hat{k}$ be the unit vector in $\vec{k}$ direction, by decomposing $\vec{x}$ into components parallel and perpendicular to $\vec{k}$, $\vec{x} = x^{\|}\,\hat{k} + \vec{x}^{\,\perp}$, the phase becomes

$$\Phi = \omega\,t - k\,x^{\|}\,, \tag{1.86}$$

where $k = |\vec{k}|$ is the length of the vector $\vec{k}$. Phases which differ by multiples of 2π give the same values for the wave W. For example, when we take $V_0 = 0$, the real part of the wave becomes

$$W_x = U_0\,\cos(\omega\,t - k\,x^{\|})$$

and $\Phi = 0,\ n\,2\pi,\ n = \pm1, \pm2, ...$ describes the wave crests. From (1.86) it follows that the crests pass by our observer with speed $\vec{u} = u\,\hat{k}$, where

$$u = \frac{\omega}{k} \ \text{ as for } \Phi = 0 \text{ we have } \quad x^{\|} = \frac{\omega}{k}\,t\,. \tag{1.87}$$

Let our observer count the number of wave crests passing by. How has then the wave (1.81) to be described in another inertial frame K'? An observer in K' who counts the number of wave crests, passing through the same space-time point at which our first observer already counts, must get the same number. After all, the coordinates are just labels and the physics is the same in all systems. When in frame K the wave takes its maximum at the space-time point (x^α) it must also be at its maximum in K' at the same space-time point in appropriately transformed coordinates (x'^α). More generally, this is holds for every value of the phase, because it is a scalar.

As (k^α) is a 4-vector the transformation law for angular frequency and wave vector is just a special case of equations (1.74), (1.75) and (1.76)

$$k'^0 = k^0 \cosh(\zeta) - k^{\|} \sinh(\zeta) = \gamma(k^0 - \beta k^{\|}) \,, \tag{1.88}$$

$$k'^{\|} = -k^0 \sinh(\zeta) + k^{\|} \cosh(\zeta) = \gamma(k^{\|} - \beta k^0) \,, \tag{1.89}$$

$$\vec{k}'^{\perp} = \vec{k}^{\perp} \,, \tag{1.90}$$

where the notation $k^{\|}$ and $k^{\perp}$ is with respect to the relative velocity of the two frames, $\vec{v}$. These transformation equations for the frequency and the wave vector describe the relativistic Doppler effect. To illustrate their meaning, let us specialize to the case of a light source, which is emitted in K and the observer K' moves in wave vector direction away from the source, i.e., $\vec{v} \parallel \vec{k}$. The equation for the wave speed (1.87) implies

$$c = \frac{\omega}{k} \;\Rightarrow\; k = |\vec{k}| = \frac{\omega}{c} = k^0$$

and chosing directions so that $k'^{\|} = k$ holds, (1.88) becomes

$$k'^0 = \gamma\,(k^0 - \beta\,k) = \gamma\,(1-\beta)\,k^0 = k^0 \sqrt{\frac{1-\beta}{1+\beta}}$$

or

$$\omega' = \frac{\nu'}{2\pi} = \omega \sqrt{\frac{1-\beta}{1+\beta}} = \frac{\nu}{2\pi} \sqrt{\frac{1-\beta}{1+\beta}} \,.$$

Now, $c = \nu\lambda = \nu'\lambda'$, where λ is the wavelength in K and λ' the wavelength in K'. Consequently, we have

$$\lambda' = \lambda \sqrt{\frac{1+\beta}{1-\beta}} \,.$$

For a receeding observer, or source receeding from the observer, $\beta > 0$ in our conventions for K and K', and the wave length λ' is larger than it is for a source at rest. This is an example of the red-shift, which is, for instance, of major importance when one analyzes spectral lines in astrophysics. Using the method of section 1, a single light signal suffices now to obtain position and speed of a distant mirror.

1.1.9 Relativistic dynamics

On a basic level this section deals with the relativistic generalization of energy, momentum and their conservation laws. So far we have introduced two units, meter to measure distances and seconds to measure time. Both are related through a fundamental constant, the speed of light, so that there is really only one independent unit up to now. In the definition of the momentum a new, independent dimensional quantity enters, the mass of a particle. This unit is defined through the gravitational law, which is out of the scope of this article. Ideally, one would like to defines mass of a body just as multiples of the mass of an elementary particle, say an electron or proton. However, this has remained too inaccurate. The mass unit has so far resisted modernization and the mass unit

$$1\ kilogram\ [kg]\ =\ 1000\ gram\ [g]$$

is still defined through a one kilogram standard object a cylinder of platinum alloy which is kept at the International Bureau of Weights and Measures at Sévres, France.

Let us consider a point-like particle in its rest-frame and denote its mass there by m_0. In any other frame the rest-mass of the particle is still m_0, which in this way is defined as a scalar. It may be noted that most books in particle and nuclear physics simply use m to denote the rest-mass, whereas many books on special relativity employ the notation to use $m = \gamma m_0$ for a mass which is proportional to the energy, *i.e.* the zero component of the energy-momentum vector introduced below. To avoid confusion, we use m_0 for the rest mass.

In the non-relativistic limit the momentum is defined by $\vec{p} = m_0\vec{u}$. We want to define $\vec{p}$ as part of a relativistic 4-vector (p^α). Consider a particle at rest in frame K, i.e., $\vec{p} = 0$. Assume now that frame K' is moving with a small velocity $\vec{v}$ with respect to K. Then the non-relativistic limit is correct, and $\vec{p}' = -m_0\vec{v}$ has to hold approximately. On the other hand, the transformation laws (1.74), (1.75) and (1.76) for vectors (note $\vec{p} \,\|\, \vec{\beta} = \vec{v}/c$) imply

$$\vec{p}\,' = \gamma\,(\vec{p} - \vec{\beta}\,p^0)\,.$$

For $\vec{p} = 0$ we find $\vec{p}\,' = -\gamma\,\vec{\beta}\,p^0)$. As in the nonrelativvstic limit $\gamma\,\beta \to \beta$, consistency requires $p^0 = c\,m_0$ in the rest frame, so that we get $\vec{p}\,' = -m_0\,\gamma\,v$. Consequently, for a particle moving with velocity $\vec{u}$ in frame K

$$\vec{p} = m_0\,\gamma\,\vec{u} \tag{1.91}$$

is the correct relation between relativistic momentum and velocity. From the invariance of the scalar product, $p_\alpha p^\alpha = (p^0)^2 - \vec{p}^2 = p'_\alpha p'^\alpha = m_0^2 c^2$ holds and

$$p^0 = +\sqrt{c^2\,m_0^2 + \vec{p}^{\,2}} \tag{1.92}$$

follows, which is of course consistent with calculating p^0 via the Lorentz transformation law (1.74). It should be noted that $c\,p^0$ has the dimension of an energy, i.e. the relativistic energy of a particle is

$$E = c\,p^0 = +\sqrt{c^4\,m_0^2 + c^2\,\vec{p}^{\,2}} = c^2\,m_0 + \frac{\vec{p}^2}{2m_0} + \ \dots\ , \tag{1.93}$$

where the second term is just the non-relativistic kinetic energy $T = \vec{p}^{\,2}/(2m_0)$. The first term shows that (rest) mass and energy can be transformed into one another [3]. In processes where the mass is conserved we just do not notice it. Using the mass definition of special relativity books like [7], $m = c\,p^0$, together with (1.93) we obtain at this point the famous equation $E = m\,c^2$. Avoiding this definition of m, because it is not the mass found in particle tables, where the mass of a particle is an invariant scalar, the essence of Einstein's equation is captured by

$$E_0 = m_0\, c^2\,,$$

where E_0 is the energy of a massive body (or particle) in its rest frame. The particle and nuclear physics literature does not use a subscript $_0$ and denotes the rest mass simply by m.

Non-relativistic momentum conservation $\vec{p}_1 + \vec{p}_2 = \vec{q}_1 + \vec{q}_2$, where $\vec{p}_i$, $(i = 1, 2)$ are the momenta of two incoming, and $\vec{q}_i$, $(i = 1, 2)$ are the momenta of two outgoing particles, becomes relativistic *energy–momentum conservation*:

$$p_1^\alpha + p_2^\alpha = q_1^\alpha + q_2^\alpha . \tag{1.94}$$

Useful formulas in relativistic dynamics are

$$\gamma = \frac{p_0}{m_0\, c} = \frac{E}{m_0\, c^2} \quad \text{and} \quad \beta = \frac{|\vec{p}|}{p^0} . \tag{1.95}$$

Further, the contravariant generalization of the velocity vector is given by

$$U^\alpha = \frac{dx^\alpha}{d\tau} = \gamma u^\alpha \;\; \text{with} \;\; u^0 = c\,, \tag{1.96}$$

compare the definition of the infinitesimal proper time (1.36). The relativistic generalization of the force is then the 4-vector

$$f^\alpha = \frac{d\,p^\alpha}{d\,\tau} = m_0\, \frac{dU^\alpha}{d\tau}\,, \tag{1.97}$$

where the last equality can only be used for particle with non-zero rest mass.

1.2 Maxwell Equations

As before all considerations are in vacuum, as for fields in a medium a preferred reference system exists. Maxwell's equations in their standard form in vacuum are

$$\nabla \vec{E} = 4\pi\rho, \quad \nabla \times \vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \frac{4\pi}{c}\vec{J}, \tag{1.98}$$

and

$$\nabla \vec{B} = 0, \quad \nabla \times \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0. \tag{1.99}$$

Here ∇ is the Nabla operator. Note that $\nabla \vec{E} = \nabla \cdot \vec{E}$, $\vec{a}\,\vec{b} = \vec{a} \cdot \vec{b}$ etc. throughout the script. $\vec{E}$ is the *electric field* and $\vec{B}$ the *magnetic field* in vacuum. When matter gets involved one introduces the *applied electric field* $\vec{D}$ and the *applied magnetic field* $\vec{H}$. Here we follow the convention of, for instance, Tipler [8] and use the notation magnetic field for the measured field $\vec{B}$, in precisely the same way as it is done for the electric field $\vec{E}$, It should be noted that this is at odds with the notation in the book by Jackson [5], where (historically correct, but quite confusingly) $\vec{H}$ is called magnetic field and $\vec{B}$ magnetic flux or magnetic induction.

Equations (1.98) are the *inhomogeneous* and equations (1.99) are the *homogeneous* Maxwell equations in vacuum.

The charge density ρ (charge per unit volume) and the current density $\vec{J}$ (charge passing through a unit area per time unit) are obviously given once a charge unit is defined through some measurement prescription. From a theoretical point of view the electrical charge unit is best defined by the magnitude of the charge of a single electron (fundamental charge unit). In more conventional units this reads

$$|q_e| = 4.80320420(19) \times 10^{-10}\, [esu] = 1.602176462(63) \times 10^{-19}\, Coulomb\, [C] \qquad (1.100)$$

where the errors are given in parenthesis. Definitions of the through measurement prescriptions rely presently on the current unit Ampére $[A]$ and are given in elementary physics textbooks like [8]. The numbers of (1.100) are from 1998 [1]. The website of the National Institute of Standards and Technology (NIST) is given in this reference. Consult it for up to date information.

The choice of constants in the inhomogeneous Maxwell equations defines units for the electric and magnetic field. The given conventions $4\pi\rho$ and $(4\pi/c)\vec{J}$ are customarily used in connection with Gaussian units, where the charge is defined in electrostatic units (esu).

In the next subsections the concepts of fields and currents are discussed in the relativistic context and the electromagnetic field equations follow in the last subsection.

1.2.1 Fields and currents

A tensor field is just a tensor function which depends on the coordinates of Minkowski space:

$$T^{\dots\alpha\dots\beta\dots} = T^{\dots\alpha\dots\beta\dots}(x)\,.$$

It is called *static* when there is no time dependence. For instance $\vec{E}(\vec{x})$ in electrostatics would be a static vector field in three dimensions. We are here, of course, primarily interested in contravariant or covariant fields in four dimensions, like vector fields $A^\alpha(x)$.

Suppose n electric charge units are contained it a small volume v, such that we can talk about the position $\vec{x}$ of this volume. The corresponding electrical charge density at the position of that volume is then just $\rho = n/v$ and the electrical current is defined as the charge that passes per unit time through a surface element of such a volume. We demand now that the electric charge density ρ and the electric current $\vec{J}$ form a 4-vector:

$$(J^\alpha) = \begin{pmatrix} c\rho \\ \vec{J} \end{pmatrix}\,.$$

Here, the factor c is introduced by dimensional reasons and we have suppressed the space-time dependence, *i.e.* $J^\alpha = J^\alpha(x)$ forms a vector field. It is left as a problem to write down the 4-current for a point particle of elementary charge q_e.

The *continuity equation* takes the simple, covariant form

$$\partial_\alpha J^\alpha = 0 \,. \tag{1.101}$$

Finally, the charge of a point particle in its rest frame is an invariant:

$$c^2 q_0^2 = J_\alpha J^\alpha \,.$$

1.2.2 The inhomogeneous Maxwell equations

The inhomogeneous Maxwell equations are obtained by writing down the simplest covariant equation which yields a 4-vector as first order derivatives of six fields. From undergraduate E&M we remember the electric and magnetic fields, $\vec{E}$ and $\vec{B}$, as the six central fields of electrodynamics. We now like to describe them in covariant form. A 4-vector is unsuitable as we like to describe six quantities E^x, E^y, E^z and B^x, B^y, B^z. Next, we may try a rank two tensor $F^{\alpha\beta}$. Then we have $4 \times 4 = 16$ quantities at our disposal. These are now too many. But, one may observe that an symmetric tensor stays symmetric under Lorentz transformation and an antisymmetric tensor stays antisymmetric. Hence, instead of looking at the full second rank tensor one has to consider its symmetric and antisymmetric parts separately.

By requesting $F^{\alpha\beta}$ to be antisymmetric,

$$F^{\alpha\beta} = -F^{\beta\alpha}, \tag{1.102}$$

this number is reduced to precisely six. The diagonal elements do now vanish,

$$F^{00} = F^{11} = F^{22} = F^{33} = 0 \,.$$

The other elements follow through (1.102) from $F^{\alpha\beta}$ with $\alpha < \beta$. As desired, this gives $(16 - 4)/2 = 6$ independent elements to start with.

Up to an over–all factor, which is chosen by convention, the only way to obtain a 4-vector through differentiation of $F^{\alpha\beta}$ is

$$\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c} J^\beta \,. \tag{1.103}$$

This is the *inhomogeneous Maxwell equation* in covariant form. Note that it determines the physical dimensions of the electric fields, the factor $c^{-1}4\pi$ on the right-hand side corresponds to Gaussian units. The continuity equation (1.101) is a simple consequence of the inhomogeneous Maxwell equation

$$\frac{4\pi}{c}\partial_\beta J^\beta = \partial_\beta \partial_\alpha F^{\alpha\beta} = 0$$

because the contraction with the symmetric tensor $(\partial_\beta \partial_\alpha)$ with the antisymmetric tensor $F^{\alpha\beta}$ is zero.

Let us choose $\beta = 0, 1, 2, 3$ and compare equation (1.103) with the inhomogeneous Maxwell equations in their standard form (1.98). For instance, $\partial_\alpha F^{\alpha\,0} = \nabla\vec{E} = 4\pi\,\rho$ yields the $F^{i\,0} = E^i$, the first column of the $F^{\alpha\beta}$ tensor. The final result is

$$(F^{\alpha\beta}) = \begin{pmatrix} 0 & -E^x & -E^y & -E^z \\ E^x & 0 & -B^z & B^y \\ E^y & B^z & 0 & -B^x \\ E^z & -B^y & B^x & 0 \end{pmatrix} . \tag{1.104}$$

Or, in components

$$F^{i0} = E^i \quad \text{and} \quad F^{ij} = -\sum_k \epsilon^{ijk}\,B^k \Leftrightarrow B^k = -\frac{1}{2}\sum_i\sum_j \epsilon^{kij}F^{ij} \ . \tag{1.105}$$

Next, $F_{\alpha\beta} = g_{\alpha\gamma}g_{\beta\delta}F^{\gamma\delta}$ implies:

$$F_{0i} = -F^{0i},\ F_{00} = F^{00} = 0,\ F_{ii} = F^{ii} = 0, \quad \text{and} \quad F_{ij} = F^{ij}.$$

Consequently,

$$(F_{\alpha\beta}) = \begin{pmatrix} 0 & E^x & E^y & E^z \\ -E^x & 0 & -B^z & B^y \\ -E^y & B^z & 0 & -B^x \\ -E^z & -B^y & B^x & 0 \end{pmatrix} . \tag{1.106}$$

1.2.3 Four-potential and homogeneous Maxwell equations

We remember that the electromagnetic fields may be written as derivatives of appropriate potentials. The only covariant option are terms like $\partial^\alpha A^\beta$. To make $F^{\alpha\beta}$ antisymmetric, we have to subtract $\partial^\beta A^\alpha$:

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha. \tag{1.107}$$

It is amazing to note that the homogeneous Maxwell equations follow now for free. The dual electromagnetic tensor is defined

$${}^*F^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta}, \tag{1.108}$$

and it holds

$$\partial_\alpha{}^*F^{\alpha\beta} = 0. \tag{1.109}$$

Proof:

$$\partial_\alpha{}^*F^{\alpha\beta} = \frac{1}{2}\left(\epsilon^{\alpha\beta\gamma\delta}\partial_\alpha\partial_\gamma A_\delta - \epsilon^{\alpha\beta\gamma\delta}\partial_\alpha\partial_\delta A_\gamma\right) = 0.$$

This first term is zero due to (1.49), because $\epsilon^{\alpha\beta\gamma\delta}$ is antisymmetric in (α, γ), whereas the derivative $\partial_\alpha\partial_\gamma$ is symmetric in (α, γ). Similarly the other term is zero. The homogeneous Maxwell equation is related to the fact that the right-hand side of equation (1.107) expresses six fields in terms of a single 4-vector. An equivalent way to write it is the equation

$$\partial^\alpha F^{\beta\gamma} + \partial^\beta F^{\gamma\alpha} + \partial^\gamma F^{\alpha\beta} = 0 \ . \tag{1.110}$$

The proof is left as an exercise to the reader.

Let us mention that the homogeneous Maxwell equation (1.109) or (1.110), and hence our demand that the field can be written in the form (1.107), excludes magnetic monopoles.

The elements of the dual tensor may be calculated from their definition (1.108). For example,

$$^*F^{02} = \epsilon^{0213} F_{13} = -F_{13} = -B^y ,$$

where the first step exploits the anti-symmetries $\epsilon^{0231} = -\epsilon^{0213}$ and $F_{31} = -F_{13}$. Calculating six components, and exploiting antisymmetry of $^*F^{\alpha\beta}$, we arrive at

$$(^*F^{\alpha\beta}) = \begin{pmatrix} 0 & -B^x & -B^y & -B^z \\ B^x & 0 & E^z & -E^y \\ B^y & -E^z & 0 & E^x \\ B^z & E^y & -E^x & 0 \end{pmatrix} . \tag{1.111}$$

The homogeneous Maxwell equations in their form (1.99) provide a non–trivial consistency check for (1.109), which is of course passed. It may be noted that, in contrast to the inhomogeneous equations, the homogeneous equations determine the relations with the $\vec{E}$ and $\vec{B}$ fields only up to an over-all $\pm$ sign, because there is no current on the right-hand side.

A notable observation is that equation (1.107) does not determine the potential uniquely. Under the transformation

$$A^\alpha \mapsto A'^\alpha = A^\alpha + \partial^\alpha \psi , \tag{1.112}$$

where $\psi = \psi(x)$ is an arbitrary scalar function, the electromagnetic field tensor is invariant: $F'^{\alpha\beta} = F^{\alpha\beta}$, as follows immediately from $\partial^\alpha \partial^\beta \psi - \partial^\beta \partial^\alpha \psi = 0$. The transformations (1.112) are called *gauge transformation*[1]. The choice of a convenient gauge is at the heart of many application.

1.2.4 Lorentz transformation for the electric and magnetic fields

The electric $\vec{E}$ and magnetic $\vec{B}$ fields are *not* components of a Lorentz four-vector, but part of the rank two the electromagnetic field $(F^{\alpha\beta})$ given by (1.104). As for any Lorentz tensor, we immediately know its behavior under Lorentz transformation

$$F'^{\alpha\beta} = a^\alpha{}_\gamma \, a^\beta{}_\delta \, F^{\gamma\delta} \, . \tag{1.113}$$

Using the explicit form (1.70) of $A = (a^\alpha{}_\beta)$ for boosts in the x^1 direction and (1.104) for the relation to $\vec{E}$ and $\vec{B}$ fields, it is left as an exercise for the reader to derive the transformation laws

$$\vec{E}\,' = \gamma \left(\vec{E} + \vec{\beta} \times \vec{B} \right) - \frac{\gamma^2}{\gamma + 1} \vec{\beta} \left(\vec{\beta}\, \vec{E} \right) \, , \tag{1.114}$$

[1] In quantum field theory these are the gauge transformations of 2. kind. Gauge transformations of 1. kind transform fields by a constant phase, whereas for gauge transformation of the 2. kind a space–time dependent function is encountered.

and

$$\vec{B}\,' = \gamma\left(\vec{B} - \vec{\beta}\times\vec{E}\right) - \frac{\gamma^2}{\gamma+1}\,\vec{\beta}\left(\vec{\beta}\,\vec{B}\right)\ . \tag{1.115}$$

1.2.5 Lorentz force

Relativistic dynamics of a point particle (more generally any mass distribution) gets related to the theory of electromagnetic fields, because an electromagnetic field causes a change of the 4-momentum of a charged particle. On a deeper level this phenomenon is related to the conservation of energy and momentum and the fact that an electromagnetic carries energy as well as momentum. Here we are content with finding the Lorentz covariant form, assuming we know already that such the approximate relationship.

We consider a charged point particle in an electromagnetic field $F^{\alpha\beta}$. Here *external* means from sources other than the point particle itself and that the influence of the point particle on these other sources (possibly causing a change of the field $F^{\alpha\beta}$) is neglected. The infinitesimal change of the 4-momentum of a point point particle is dp^α and assumed to be proportional to (i) its charge q and (ii) the external electromagnetic field $F^{\alpha\beta}$. This means, we have to contract $F^{\alpha\beta}$ with some infinitesimal covariant vector to get dp^α. The simplest choice is dx_β, what means that the amount of 4-momentum change is proportional to the space-time length at which the particle experiences the electromagnetic field. Hence, we have determined dp^α up to a proportionality constant, which depends on the choice of units. Gaussian units are defined by choosing c^{-1} for this proportionality constant and we have

$$dp^\alpha = \pm\frac{q}{c}\,F^{\alpha\beta}\,dx_\beta. \tag{1.116}$$

As discussed in the next section, it is a consequence of energy conservation, in this context known as Lenz's law, that the force between charges of equal sign has to be repulsive. This corresponds to the plus sign and we arrive at

$$dp^\alpha = \frac{q}{c}\,F^{\alpha\beta}\,dx_\beta. \tag{1.117}$$

Experimental measurements are of course in agreement with this sign. The remarkable point is that energy conservation and the general structure of the theory already imply that the force between charges of equal sign has to be *repulsive*. Therefore, despite the similarity of the Coulomb's inverse square force law with Newton's law it impossible to build a theory of gravity along the lines of this chapter, *i.e.* to use the 4-momentum p^α as source in the inhomogeneous equation (1.103). The resulting force would necessarily be repulsive. Experiments show also that positive and negative electric charges exist and deeper insight about their origin comes from the relativistic Lagrange formulation, which ultimately has to include Dirac's equation for electrons and leads then to Quantum Electrodynamics.

Taking the derivative with respect to the proper time, we obtain the 4-force acting on a charged particle, called *Lorentz force*,

$$f^\alpha = \frac{dp^\alpha}{d\tau} = \frac{q}{c}F^{\alpha\beta}U_\beta. \tag{1.118}$$

As in equation (1.97) $f^\alpha = m_0\, dU^\alpha/d\tau$ holds for non-zero rest mass and the definition of the contravariant velocity is given by equation (1.96).

Using the representation (1.104) of the electromagnetic field the time component of the relativistic Lorentz force, which describes the change in energy, is

$$f^0 = \frac{dp^0}{d\tau} = -\frac{q}{c}\left(\vec{E}\vec{U}\right) \ . \tag{1.119}$$

To get the space component of the Lorentz force we use besides (1.104) equation (1.105) which give the equality

$$\frac{q}{c}\sum_{j=1}^{3} F^{ij} U_j = -\frac{q}{c}\sum_{j=1}^{3}\sum_{k=1}^{3} \epsilon^{ijk} B^k U_j$$

The space components combine into the well-known equation

$$\vec{f} = q\,\gamma\,\vec{E} + \frac{q}{c}\,\vec{U}\times\vec{B} \tag{1.120}$$

where our derivation reveals that the relativistic velocity (1.96) of the charge q and not its velocity $\vec{v}$ enters the force equation. This allows, for instance, correct force calculations for fast flying electrons in a magnetic field. The equation (1.120) for $\vec{f}$ may now be used to define a measurement prescription for an electric charge unit.

1.3 Faraday's Law

From the homogeneous Maxwell equations (1.109) we have

$$\nabla\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0\,. \tag{1.121}$$

This equation is the *differential form of Faraday's law*: A changing magnetic field induces an electric field. In the following we derive the integral form, which is needed for circuits of macroscopic extensions. We integrate over a simply connected surface S and use Stoke's theorem to convert the integral over $\nabla\times\vec{E}$ into a closed line integral along the boundary C of S:

$$\int_S (\,\nabla\times\vec{E}\,)\cdot d\vec{a} = \oint_C \vec{E}\cdot d\vec{l} = -\frac{1}{c}\int_S \frac{\partial\vec{B}}{\partial t}\cdot d\vec{a}\,.$$

On the right-hand side we eliminate the partial derivative $\partial/\partial t$ using

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{v}\cdot\nabla \qquad \left(\text{note } \ \vec{v} = \frac{\partial\vec{x}}{\partial t} = \sum_{i=1}^{3}\hat{e}^i\,\frac{\partial x^i}{\partial t}\right)$$

to get

$$\oint_C \vec{E}\cdot d\vec{l} = -\frac{1}{c}\frac{d}{dt}\int_S \vec{B}\cdot d\vec{a} + \frac{1}{c}\int_S (\vec{v}\cdot\nabla)\,\vec{B}\cdot d\vec{a} \tag{1.122}$$

Using the other homogeneous Maxwell equation, $\nabla\cdot\vec{B}=0$, and that the $\partial/\partial x^i$ derivatives of $\vec{v}$ vanish (e.g., $(\partial/\partial x^1)\,(\partial x^1/\partial t) = (\partial/\partial t)\,(\partial x^1/\partial x_1) == 0$), the well-known vector identity

$$[\nabla\times(\,\vec{a}\times\vec{b}\,)] = (\,\nabla\cdot\vec{b}\,)\,\vec{a} + (\,\vec{b}\cdot\nabla\,)\,\vec{a} - (\,\nabla\cdot\vec{a}\,)\,\vec{b} - (\,\vec{a}\cdot\nabla\,)\,\vec{b}\,,$$

gives

$$\nabla\times\left(\vec{B}\times\vec{v}\right) = \vec{v}\,\left(\nabla\cdot\vec{B}\right) + (\vec{v}\cdot\nabla)\,\vec{B} = (\vec{v}\cdot\nabla)\,\vec{B}$$

and we transform the last integral in (1.122) as follows:

$$\begin{aligned}\frac{1}{c}\int_S(\vec{v}\cdot\nabla)\,\vec{B}\cdot d\vec{a} &= -\frac{1}{c}\int_S\nabla\times\left(\vec{v}\times\vec{B}\right)\cdot d\vec{a} = \\ -\frac{1}{c}\oint_C\left(\vec{v}\times\vec{B}\right)\cdot d\vec{l} &= -\oint_C\left(\vec{\beta}\times\vec{B}\right)\cdot d\vec{l}\end{aligned}$$

where Stoke's theorem has been used and $\vec{\beta}=\vec{v}/c$. We re-write equation (1.122) with both encountered line integral on the left-hand side

$$\oint_C\left(\vec{E}+\vec{\beta}\times\vec{B}\right)\cdot d\vec{l} = -\frac{1}{c}\frac{d}{dt}\Phi_m \tag{1.123}$$

where Φ is called *magnetic flux* and defined by

$$\Phi_m = \int_S\vec{B}\cdot d\vec{a}\;. \tag{1.124}$$

Equation (1.123) is the fully relativistic version of Faraday's law. Let us discuss the approximations which lead to its original, non-relativistic version. The velocity $\vec{\beta}=\vec{v}/c$ in equation (1.123) refers to the velocity of the line element $d\vec{l}$ with respect to the inertial frame in which the calculation is done. Let us consider a particular line element $d\vec{l}$ and transfer the electric field to a frame co-moving with this line element. Equation (1.114) yields

$$\vec{E}\,' = \gamma\,\left(\vec{E}+\vec{\beta}\times\vec{B}\right) - \frac{\gamma^2}{\gamma+1}\,\vec{\beta}\,\left(\vec{\beta}\cdot\vec{B}\right) \to \vec{E}+\vec{\beta}\times\vec{B}\;\;\text{for}\;\;|\vec{v}|\ll c$$

and in this approximation we have

$$\epsilon_{\rm emf} = \oint_C\vec{E}'\cdot d\vec{l} = -\frac{1}{c}\frac{d}{dt}\Phi_m \tag{1.125}$$

where $\epsilon_{\rm emf}$ is called *electromotive force* (emf) and the electric field $\vec{E}\,'$ is with respect to the frames co-moving with $d\vec{l}$. If the velocity difference between the involved line elements are small one may define an appropriate rest frame, normally the Lab frame, for the entire current loop and the field $\vec{E}_{\rm lab}$ in this frame is a good approximation for $\vec{E}'$ in (1.125). In this approximation *Faraday's Law of Induction* is found in most test books. Due to our initial treatment of special relativity we do not face the problem to work out its relativistic generalization, but instead obtained (1.125) as the limit of the generally correct law (1.123).

1.3.1 Lenz's law

With the Lorentz force (1.118) given, Energy conservation determines the minus sign on the right-hand side of Faraday's law (1.125). This is known as *Lenz's law.* For closed, conducting circuits the emf (1.125) will induce a current, whose magnitude depends on the resistance of the circuit. Lenz's law states: *The induced emf and induced current are in such a direction as to oppose the change that produces them.* [8] gives many examples. To illustrate the connection with energy conservation, we discuss one of them.

We consider a permanent bar magnet moving towards a closed loop that has a resistance R. The north pole of the bar magnet is defined so that the magnetic field points out of it. We arrange the north–south axis of the magnet perpendicular to the surface spanned by the loop and move the magnet toward the loop. The magnet's magnetic field through the loop get stronger when the magnet is approaching and a current is induced in the loop. The direction of the current is such that its magnetic field is opposite to that of the magnet, effectively the loop becomes a magnet with north pole towards the bar magnet. The result is a *repulsive* force between bar magnet and loop. Work against this force is responsible for the induced current, and its associated heat, in the loop. Would the sign of the induced current be different an attractive force would result and the resulting acceleration of the bar magnet as well as the heat in the loop would violate energy conservation. Note that pulling the bar magnet out of the loop does also produce energy.

In our treatment the sign of Faraday's law is already given by the electromagnetic field equation and energy conservation determines the sing in equation (1.116) for the Lorenz force.

1.3.2 LCR circuit

We consider a resistor (R), a capacitor (C) and an inductor (L) in series, see figure 1.2. Charge conservation implies

$$I_R \;=\; I_C \;=\; I_L \;= I \tag{1.126}$$

where I_R, I_C and I_L are the currents at the resistor, capacitor and inductor, respectively. The voltages are related to the currents by the following equations

$$V_R \;=\; R\,I \;\text{ and }\; V_C \;=\; \frac{Q}{C} \;\text{ with }\; \frac{dQ}{dt} \;=\; I \tag{1.127}$$

where $Q(t)$ is the charge on the capacitor. As inductor we consider an idealized infinitely long solenoid. The induced (back) electromotive force (voltage over the inductor) is the derivative of the magnetic flux (1.124)

$$\epsilon_{\text{back}} \;=\; V_L \;=\; \frac{1}{c}\frac{d\Phi_m}{dt} \;=\; L\,\frac{dI}{dt} \tag{1.128}$$

where the constant $L > 0$ is called *self inductance* of the coil. In accordance with Lenz's law the induced electromotive ϵ_{back} will work against the electromotive force ϵ_{emf} which drives

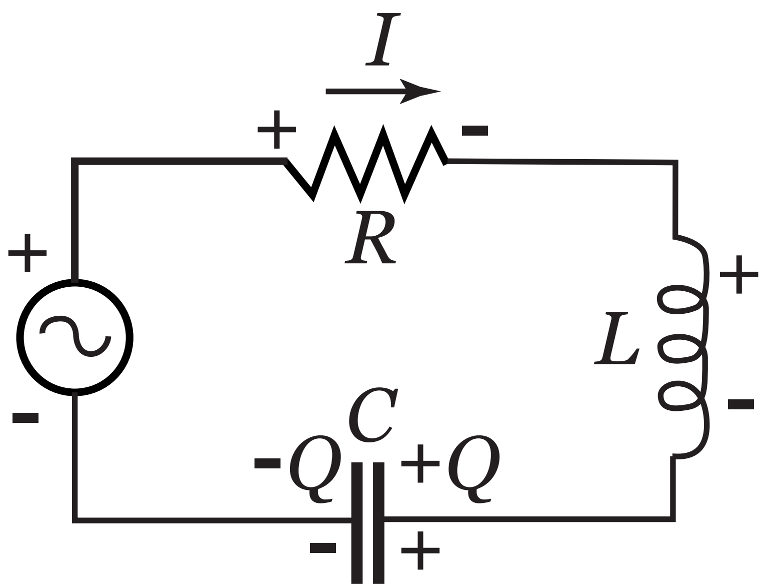

Figure 1.2: Series LCR circuit with an ac generator.

the circuit. We use complex calculus and the physical quantities are given by the real parts. For the total voltage we have

$$\epsilon_{\rm emf} \;=\; \epsilon_{\rm max}\, e^{-i\omega t} \;=\; V \;=\; V_L + V_R + V_C \tag{1.129}$$

where ω is the driving frequency. We replace in this relation the voltages by their equations listed above and eliminate the current I in favor of the time derivative of the charge of the capacitor and to arrive at the second order differential equation

$$L\,\frac{d^2Q}{dt^2} + R\,\frac{dQ}{dt} + \frac{Q}{C} \;=\; \epsilon_{\rm max}\, e^{-i\omega t}\ . \tag{1.130}$$

The general solution of this equation is obtained as superposition of a special solution of the inhomogenous equation and the general solution of the homogeneous equation, $Q = Q_h + Q_i$. A special solution of the inhomogeneous equation is

$$Q_i \;=\; Q_0\, e^{-i\omega t} \tag{1.131}$$

where

$$\left(-\omega^2\, L - i\,\omega\, R + \frac{1}{C}\right)\, Q_0 = \epsilon_{\rm max} \quad\Rightarrow\quad Q_0 = \frac{\epsilon_{\rm max}}{-\omega^2\, L - i\,\omega\, R + 1/C}\ . \tag{1.132}$$

The general solution of the homogeneous equation follows from the ansatz $Q_h \sim e^{i\alpha t}$, which implies the algebraic equation

$$-\alpha^2\, L + i\,\alpha\, R + C^{-1} \;=\; 0\ .$$

The solution is

$$\alpha \;=\; i\,\frac{R}{2L} \pm \omega_h \;\text{ with }\; \omega_h = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \tag{1.133}$$

and we arrive at the general solution of the homogeneous equation

$$Q_h \;=\; A_1\, e^{-tR/(2L)}\, e^{+i\omega_h t} + A_2\, e^{-tR/(2L)}\, e^{-i\omega_h t}\;. \tag{1.134}$$

Due to the exponential damping the homogeneous solution dies out for large times and we are left with the inhomogeneous solution $Q = Q_i$ (1.131). The homogeneous solution has to be used for getting the initial values right, e.g. for turning the circuit on or off. In the large time limit the essential features of the LCR circuit are those of the inhomogeneous solution, which we investigate now in more details.

For the inhomogeneous solution (1.131) Q_0 is a complex constant. To separate the amplitude of Q_0 from its phase we write

$$Q_0 \;=\; Q_{\max}\, e^{-i\delta} \;\text{ with }\; Q_{\max} \;=\; |Q_0| \;=\; \frac{\epsilon_{\max}}{\sqrt{(1/C - \omega^2 L)^2 + \omega^2 R^2}}\;. \tag{1.135}$$

Following standard conventions, we rewrite the expression for $Q_{\max}$ as

$$Q_{\max} \;=\; \frac{\epsilon_{\max}}{\omega\sqrt{[1/(\omega C) - \omega L]^2 + R^2}} \;=\; \frac{\epsilon_{\max}}{\omega\sqrt{(X_C - X_L)^2 + R^2}} \;=\; \frac{\epsilon_{\max}}{\omega Z}\;. \tag{1.136}$$

Here

$$X_C \;=\; \frac{1}{\omega C} \;\text{ is called } \textit{capacitive reactance}, \tag{1.137}$$

$$X_L \;=\; \omega L \;\text{ is called } \textit{inductive reactance}, \tag{1.138}$$

and the difference $X_L - X_C$ is called *total reactance*. The quantity

$$Z = \sqrt{(X_C - X_L)^2 + R^2} \;\text{ is called } \textit{impedance}. \tag{1.139}$$

Using this notation the original equation (1.132) for Q_0 becomes

$$Q_0 \;=\; \frac{\epsilon_{\max}}{\omega\,(X_C - X_L - iR)} \;=\; \frac{\epsilon_{\max}\,(X_C - X_L + i\,R)}{\omega Z^2}\;. \tag{1.140}$$

It follows from $Q_0 = Q_{\max}\cos(\delta) - i\,Q_{\max}\sin(\delta)$ that the phase δ is given by[1]

$$\tan\delta \;=\; -\frac{\sin(\delta)}{\cos(\delta)} \;=\; -\frac{\mathrm{Im}\, Q_0}{\mathrm{Re}\, Q_0} \;=\; \frac{R}{X_L - X_C}\;. \tag{1.141}$$

From equation (1.127) we see that V_C is in phase with the charge Q

$$V_C = \frac{Q_{\max}}{C}\, e^{-i\delta}\, e^{-i\omega t} \tag{1.142}$$

[1]Note that in [8] the phase δ is defined with respect to the current instead of the charge, what leads to the equation $\delta = (X_L - X_C)/R$.

and the phase of the current I is shifted by $\pi/2$ with respect to V_C:

$$I = \frac{dQ}{dt} = -i\,\omega\,Q_{\rm max}\,e^{-i\delta}\,e^{-i\omega t} = \omega\,Q_{\rm max}\,e^{-i\delta-i\pi/2}\,e^{-i\omega t} = I_{\rm max}\,e^{-i\delta-i\pi/2}\,e^{-i\omega t} \tag{1.143}$$

where we used $-i = \exp(-i\pi/2)$. Note also that $I_{\rm max} = \omega\,Q_{\rm max}$ together with equation (1.136) implies

$$I_{\rm max} = \frac{\epsilon_{\rm max}}{Z} = \frac{\epsilon_{\rm max}}{\sqrt{[1/(\omega\,C) - \omega\,L]^2 + R^2}}\ . \tag{1.144}$$

The voltage over the resistor is in phase with the current

$$V_R = \frac{I_{\rm max}}{R}\,e^{-i\delta-i\pi/2}\,e^{-i\omega t} \tag{1.145}$$

whereas the voltage over the inductor is shifted by another $\pi/2$ phase factor:

$$V_L = L\,\frac{dI}{dt} = \omega\,L\,I_{\rm max}\,e^{-i\delta t-i\pi}\,e^{i\omega t}\ . \tag{1.146}$$

The current amplitude (1.144) takes its maximum at the *resonance frequency*

$$\omega_0 = \frac{1}{\sqrt{LC}}\ . \tag{1.147}$$

Note that $Q_{\rm max}(\omega_0) = \epsilon_{\rm max}/(\omega_0 R)$ due to equation (1.136). For $R \neq 0$ the maximum of $Q_{\rm max}$ is shifted by an amount of order $(R/L)^2$ away from ω_0. We find it by differentiation of $Q_{\rm max}$ with respect to ω^2:

$$-2\,L\,(C^{-1} - \omega_{\rm max}^2\,L) + R^2 = 0 \implies \omega_{\rm max} = \sqrt{\frac{1}{LC} - \frac{R^2}{2\,L^2}}\ . \tag{1.148}$$

Note, that this $\omega_{\rm max}$ value is lower than the frequency ω_h of the damped homogeneous solution (1.134), while the resonance frequency ω_0 is larger.

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