FINAL ADVANCED DYNAMICS – PHY 4241 (Spring, 2010)

PROBLEM 1 (30 points)

Assume a Lagrangian $L = L(\{q_i\}, \{\dot{q}_i\}, t)$ where q_i, $i = 1, \ldots, n$ are generalized coordinates, $\dot{q}_i$, $i = 1, \ldots, n$ are generalized velocities and t is the time.

1. Write down the principle of least action.
2. Derive the Euler-Lagrange equations from the principle of least action.
3. Assume that the Lagrangian is invariant under translations $q_i \to q_i' = q_i + \epsilon_i$. Find the corresponding conserved quantities.

PROBLEM 2 (30 points)

The electromagnetic field tensor is given by

$$(F^{\alpha\beta}) = \begin{pmatrix} 0 & -E^x & -E^y & -E^z \\ E^x & 0 & -B^z & B^y \\ E^y & B^z & 0 & -B^x \\ E^z & -B^y & B^x & 0 \end{pmatrix} .$$

Find (the Einstein summation convention is assumed for identical indices):

1. $F^{\alpha}{}_{\alpha}$.
2. The matrix $(F_{\alpha\beta})$.
3. $F_{\alpha\beta}F^{\alpha\beta}$.

PROBLEM 3 (40 points)

Let the interaction of two point particles be described by the Lagrangian

$$L = \frac{m_1}{2}\dot{\vec{r}}_1^{\,2} + \frac{m_2}{2}\dot{\vec{r}}_2^{\,2} - \frac{\alpha}{|\vec{r}_1 - \vec{r}_2|} .$$

1. Is the energy of this system conserved (with reason)?
2. Write down the definition of the center of mass $\vec{R}(t)$ of the system. How is the center of mass reference frame defined?
3. Let $\vec{r} = \vec{r}_1 - \vec{r}_2$. Express the Lagrangian in the center of mass frame as function of the coordinate $\vec{r}$ and velocity $\dot{\vec{r}}$.
4. Rewrite the resulting Lagrangian in spherical coordinates (polar angle θ, azimuth angle ϕ, and magnitude $r = |\vec{r}|$).
5. Write down the Euler-Lagrange equations for the two angles and identify a conserved quantity.
6. Find $\phi(t)$ for the solution with the initial condition $\dot{\phi}(0) = 0$ where ϕ is the azimuth angle.
7. Can one always find a coordinate system so that $\dot{\phi}(0) = 0$ holds?