FINAL ADVANCED DYNAMICS – PHY 4241/5227 (April 29, 2009)

PROBLEM 1 (30 points)

Assume a Lagrangian $L = L(\{q_i\}, \{\dot{q}_i\}, t)$ where q_i, $i = 1, \ldots, n$ are generalized coordinates, $\dot{q}_i$, $i = 1, \ldots, n$ are generalized velocities and t is the time.

1. Write down the principle of least action.
2. Derive the Euler-Lagrange equations from the principle of least action.
3. Assume that the Lagrangian is invariant under translations $q_i \to q_i' = q_i + \epsilon_i$. Find the corresponding conserved quantities.

PROBLEM 2 (30 points)

An electromagnetic field tensor (in arbitrary units) is given by

$$(F^{\alpha\beta}) = \begin{pmatrix} 0 & 1 & 2 & -1 \\ -1 & 0 & -2 & 1 \\ -2 & 2 & 0 & 3 \\ 1 & -1 & -3 & 0 \end{pmatrix} .$$

Find (the Einstein summation convention is assumed for identical indices):

1. $F^{\alpha}{}_{\alpha}$.
2. The matrix $(F_{\alpha\beta})$.
3. $F_{\alpha\beta}F^{\alpha\beta}$.

PROBLEM 3 (40 points)

1. Write down the three components, E^1, E^2 and E^3 of the electric field of a static point charge q, which is located at the origin of a coordinate system described by x^0, x^1, x^2 and x^3.
2. What is the magnetic field $\vec{B}$ of this point charge?
3. Write the down the electromagnetic field tensor $(F^{\alpha\beta})$ of this point charge.
4. Use the transformation law $F'(x) = A\,F(x)\,A^{\dagger}$ and the Lorentz matrix

$$A = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

   to calculate the electromagnetic field tensor $(F'^{\alpha\beta})$ for an inertial frame K', which moves with velocity v along the x-axis of the rest frame of the point charge (the orientations of the axis of the two frames are parallel and their origins coincide at time $t = t' = 0$).
5. The previous expression $F'(x)$ is still written in x-coordinates. Use now x'-coordinates to write down the electric $E'^{i}(x')$ and magnetic $B'^{i}(x')$, $i = 1, 2, 3$ field components in the frame K'.