SOLUTIONS FINAL ADVANCED DYNAMICS

PHY 4241 (Spring, 2010)

PROBLEM 1

See Homework 6 and $L(q_i + \epsilon_i, \dot{q}_i, t) = L(q_i, \dot{q}_i, t)$ implies

$$0 = \frac{\partial L}{\partial q_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} \;\Rightarrow\; p_i = \frac{\partial L}{\partial \dot{q}_i} \text{ conserved (generalized momentum).}$$

PROBLEM 2

The electromagnetic field tensor is given by Find (the Einstein summation convention is assumed for identical indices):

1. ${F^\alpha}_\alpha = 0$.
2. $$(F_{\alpha\beta}) = \begin{pmatrix} 0 & +E^x & +E^y & +E^z \\ -E^x & 0 & -B^z & B^y \\ -E^y & B^z & 0 & -B^x \\ -E^z & -B^y & B^x & 0 \end{pmatrix} .$$
3. $F_{\alpha\beta}F^{\alpha\beta} = 2\left(\vec{B}^{\,2} - \vec{E}^{\,2}\right)$.

PROBLEM 3

1. Yes, because L does not depend explicitly on time.
2. $$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} .$$
   The center of mass frame is defined by $\vec{R} = 0$.
3. $$\vec{r}_1 - \vec{r}_2 + \left(\frac{m_1 + m_2}{m_2}\right)\vec{R} = +\vec{r}_1 + \frac{m_1}{m_2}\vec{r}_1 = \frac{m_1 + m_2}{m_2}\vec{r}_1 \,,$$
   $$\vec{r}_1 - \vec{r}_2 - \left(\frac{m_1 + m_2}{m_1}\right)\vec{R} = -\vec{r}_2 - \frac{m_2}{m_1}\vec{r}_2 = \frac{m_1 + m_2}{m_1}\vec{r}_2 \,,$$
   Therefore, in the center of mass frame
   $$\vec{r}_1 = \frac{m_2\,\vec{r}}{m_1 + m_2} \quad \text{and} \quad \vec{r}_2 = -\frac{m_1\,\vec{r}}{m_1 + m_2} .$$
   The Lagrangian becomes
   $$L = \mu\,\dot{\vec{r}}^{\,2} - \frac{\alpha}{|\vec{r}|} \;\text{ with }\; \mu = \frac{m_1 m_2}{m_1 + m_2} \;\text{ reduced mass}.$$

4. In spherical coordinates we have

$$L = \frac{1}{2}\,\mu\,\dot{r}^2 + \frac{1}{2}\,\mu\,r^2\,\left(\dot{\theta}^2 + \sin^2(\theta)\,\dot{\phi}^2\right) - \frac{\alpha}{|\vec{r}|}\,.$$

5. The Euler-Lagrange equations for the angles give

$$0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 2\mu\,r\,\dot{r}\,\dot{\theta} + \mu\,r^2\,\left(\ddot{\theta} - \sin(\theta)\,\cos(\theta)\,\dot{\theta}\,\dot{\phi}^2\right)\,,$$

$$0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi} = \frac{d}{dt}\left(\frac{1}{2}\,\mu\,r^2\,\sin^2(\theta)\,\dot{\phi}\right)\,.$$

From the last equation we find that angular momentum is conserved:

$$L_\phi = \mu\,r^2\,\sin^2(\theta)\,\dot{\phi} = \text{constant}\,.$$

6. Let $\dot{\phi}(0) = 0$ and $\theta \neq 0,\, \pi$. Then for *all* times

$$\sin^2[\theta(t)]\,\dot{\phi}(t) = 0 \;\Rightarrow\; \dot{\phi}(t) = 0 \;\Rightarrow\; \phi(t) = \text{constant}\,.$$

The motion is in a plane.

7. Yes, just choose $\hat{\phi}(0)$ orthogonal to $\vec{r}(0) \times \dot{\vec{r}}(0)$.