SOLUTIONS MIDTERM ADVANCED DYNAMICS — PHY–4241/5227

PROBLEM 1 (20 points)

1.
$$\frac{\partial L}{\partial q_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\,, \quad p_i = \frac{\partial L}{\partial \dot{q}_i}\,, \quad \text{for} \quad i = 1, \dots, n\,.$$

2.
$$L = \frac{1}{2}\, m\, \dot{x}^2 - \frac{1}{2}\, k\, x^2\,.$$

3.
$$p = \frac{\partial L}{\partial \dot{x}} = m\, \dot{x}\,.$$

4.
$$H = \frac{p^2}{2m} + \frac{1}{2}\, k\, x^2\,.$$

5.
$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m}\,, \qquad -\,\dot{p} = \frac{\partial H}{\partial x} = k\, x\,.$$

6. Combining the previous two equations we find

$$\dot{p} = m\, \ddot{x} = -k\, x\,.$$

PROBLEM 2 (20 points)

1.
$$L = \frac{1}{2}\, m\, \dot{r}^2 + \frac{1}{2}\, m\, r^2\, \left(\dot{\theta}^2 + \sin^2(\theta)\, \dot{\phi}^2\right) - V(r)\,.$$

2.
$$0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 2m\, r\, \dot{r}\, \dot{\theta} + m\, r^2\, \left(\ddot{\theta} - \sin(\theta)\, \cos(\theta)\, \dot{\theta}\, \dot{\phi}^2\right)\,,$$
$$0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi} = \frac{1}{2}\, \frac{d}{dt}\left(m\, r^2\, \sin^2(\theta)\, \dot{\phi}\right)\,.$$

3. From the last equation we find that angular momentum is conserved:

$$L_\phi = m\, r^2\, \sin^2(\theta)\, \dot{\phi} = \text{constant}\,.$$

As the Lagrangian L does not explicitly depend on time, energy

$$E = \sum_i \frac{\partial L}{\partial \dot{q}_i}\, \dot{q}_i - L$$

is conserved. Here

$$E = \frac{1}{2}\, m\, \dot{r}^2 + \frac{1}{2}\, m\, r^2\, \left(\dot{\theta}^2 + \sin^2(\theta)\, \dot{\phi}^2\right) + V(r)\,.$$

4. Let $\dot{\phi}(0) = 0$ and $\theta \neq 0, \pi$. Then for *all* times

$$\sin^2[\theta(t)]\,\dot{\phi}(t) = 0 \;\Rightarrow\; \dot{\phi}(t) = 0 \;\Rightarrow\; \phi(t) = \text{constant}\,.$$

The motion is in a plane.

PROBLEM 3 (10 points)

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}\,, \qquad \nabla \cdot \vec{B} = 0\,,$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}\,, \qquad \nabla \times \vec{B} = \mu_0\,\vec{J} + \mu_0\,\epsilon_0\,\frac{\partial \vec{E}}{\partial t}\,.$$