ADVANCED DYNAMICS — PHY 4241/5227
HOME AND CLASS WORK – SET 3

Solution for assignment 13: Double Pendulum.

Derivation of the Lagrangian:

$$
\begin{aligned}
x_1 &= l\,\sin\phi \\
y_1 &= l\,\cos\phi \\
x_2 &= l\,\sin\phi + l\,\sin\psi \\
y_2 &= l\,\cos\phi + l\,\cos\psi \\
L &= \frac{1}{2}\,m\,l\,\left(\dot{x_1}^2 + \dot{y_1}^2\right) + \frac{1}{2}\,m\,l\,\left(\dot{x_2}^2 + \dot{y_2}^2\right) - m\,g\,l\,(z_1 + z_2) + \text{const} \\
&= \frac{1}{2}\,m\,l\,\dot\phi^2 + \frac{1}{2}\,m\,l\,\left(\dot\phi^2 + \dot\psi^2 + 2\,\dot\phi\,\dot\psi\,\cos(\phi-\psi)\right) + m\,g\,l\,\left(2\,\cos\phi + \cos\psi\right) \\
&= ,m\,l\,\left(\dot\phi^2 + \frac{1}{2}\,\dot\psi^2 + \dot\phi\,\dot\psi\,\cos(\phi-\psi)\right) + m\,g\,l\,\left(2\,\cos\phi + \cos\psi\right)\ .
\end{aligned}
$$

Taylor expansion to second order in ϕ, ψ, $\dot\phi$ and $\dot\psi$ gives

$$L = m\,l\,\left(\dot\phi^2 + \frac{1}{2}\,\dot\psi^2 + \dot\phi\,\dot\psi\right) - m\,g\,l\,\left(\phi^2 + \frac{1}{2}\,\psi^2\right) + 3\,m\,g\,l\ .$$

The Euler-Lagrange equation are then

$$
\begin{aligned}
0 &= \frac{d}{dt}\frac{\partial L}{\partial\dot\phi} - \frac{\partial L}{\partial\phi} &= 2\,\ddot\phi + \ddot\psi + 2\,\frac{g}{l}\,\phi \\
0 &= \frac{d}{dt}\frac{\partial L}{\partial\dot\psi} - \frac{\partial L}{\partial\psi} &= \ddot\psi + \ddot\phi + \frac{g}{l}\,\psi
\end{aligned}
$$

and in matrix notation

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \ddot\phi \\ \ddot\psi \end{pmatrix} + \begin{pmatrix} 2g/l & 0 \\ 0 & g/l \end{pmatrix}\begin{pmatrix} \phi \\ \psi \end{pmatrix} = 0\ .$$

This is solved by the exponential ansatz (physical is the real part of the solution):

$$\Phi(t) = \begin{pmatrix} \phi \\ \psi \end{pmatrix} = e^{i\,\omega\,t}\begin{pmatrix} \phi_0 \\ \psi_0 \end{pmatrix} \quad\Rightarrow\quad \begin{pmatrix} \ddot\phi \\ \ddot\psi \end{pmatrix} = -\omega^2\,e^{i\,\omega\,t}\begin{pmatrix} \phi_0 \\ \psi_0 \end{pmatrix}$$

$$\begin{pmatrix} -2\omega^2 + 2g/l & -\omega^2 \\ -\omega^2 & -\omega^2 + g/l \end{pmatrix}\begin{pmatrix} \phi \\ \psi \end{pmatrix} = 0\ .$$

$$0 = \det\begin{vmatrix} -2\omega^2 + 2g/l & -\omega^2 \\ -\omega^2 & -\omega^2 + g/l \end{vmatrix} = \omega^4 - \frac{4g}{l}\,\omega^2 + \frac{2g^2}{l^2}$$

with eigenmodes

$$\omega_\pm = \sqrt{\frac{g}{l}}\,\sqrt{2 \pm \sqrt{2}}\ .$$