ADVANCED DYNAMICS — PHY 4241/5227

HOME AND CLASS WORK – SET 3

(January 20, 2010)

Solution for assignment 14: Energy conservation.

The total time derivative of the Lagrangian $L(\dot{q}_j, q_j, t)$ is

$$\frac{dL}{dt} \;=\; \sum_j \left(\frac{\partial L}{\partial \dot{q}_j} \frac{d\dot{q}_j}{dt} + \frac{\partial L}{\partial q_j} \frac{dq_j}{dt} \right) + \frac{\partial L}{\partial t} \;.$$

By its definition $\delta_t L = (\partial L/\partial t)\, \delta t = 0$ implies $(\partial L/\partial t) = 0$, so that

$$\frac{dL}{dt} \;=\; \sum_j \left(\frac{\partial L}{\partial \dot{q}_j}\, \ddot{q}_j + \frac{\partial L}{\partial q_j}\, \dot{q}_j \right)$$

holds. Using the Euler-Lagrange equation, we find the desired result:

$$\frac{dL}{dt} \;=\; \sum_j \left(\ddot{q}_j \frac{\partial L}{\partial \dot{q}_j} + \dot{q}_j \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} \right) = \frac{d}{dt} \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} \quad \Leftrightarrow \quad \frac{d}{dt} \left(L - \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} \right) = 0 \;.$$

For a bilinear kinetic Energy we find by differentiation and carrying the sum over i out

$$\begin{aligned} \sum_i \dot{q}_i \frac{\partial T}{\partial \dot{q}_i} \;&=\; \sum_i \dot{q}_i \frac{\partial}{\partial \dot{q}_i} \sum_{j,k} a_{jk}\, \dot{q}_j \dot{q}_k \;=\; \sum_i \dot{q}_i \sum_{j,k} a_{jk}\, (\delta_{ij}\, \dot{q}_k + \dot{q}_j\, \delta_{ik}) \\ &=\; \sum_{j,k} a_{jk}\, (\dot{q}_j\, \dot{q}_k + \dot{q}_j\, \dot{q}_k) \;=\; 2\,T \;. \end{aligned}$$