ADVANCED DYNAMICS — PHY 4241/5227

HOME AND CLASS WORK – SET 4

Solution for assignment 18: Legendre transformation and Hamilton's equations for the spherical pendulum.

(a) Legendre transformation. With

$$L = T - V \quad \text{with} \quad T = \frac{1}{2}\, m\, b^2\, \dot\theta^2 + \frac{1}{2}\, m\, b^2\, (\sin\theta)^2\, \dot\phi^2 \quad \text{and} \quad V = -m\, g\, b\, \cos\theta$$

the generalized momenta are

$$p_\theta = \frac{\partial L}{\partial \dot\theta} = m\, b^2\, \dot\theta \quad \text{and} \quad p_\phi = \frac{\partial L}{\partial \dot\phi} = m\, b^2\, (\sin\theta)^2\, \dot\phi\ .$$

Therefore,

$$\dot\theta\, p_\theta + \dot\phi\, p_\phi = 2T \quad \text{and} \quad H = \dot\theta\, p_\theta + \dot\phi\, p_\phi - L = \frac{1}{2}\, (\dot\theta\, p_\theta + \dot\phi\, p_\phi) = T + V\ .$$

Eliminating

$$\dot\theta = \frac{p_\theta}{m\, b^2} \quad \text{and} \quad \dot\phi = \frac{p_\phi}{m\, b^2\, (\sin\theta)^2}\ .$$

from H, we have

$$H = \frac{p_\theta^2}{2\, m\, b^2} + \frac{p_\phi^2}{2\, m\, b^2\, (\sin\theta)^2} - m\, g\, b\, \cos\theta\ .$$

(b) Hamilton's equations:

$$\begin{aligned}
\dot\theta &= \frac{\partial H}{\partial\, p_\theta} = \frac{p_\theta}{m\, b^2}\,, \\
\dot\phi &= \frac{\partial H}{\partial\, p_\phi} = \frac{p_\phi}{m\, b^2\, (\sin\theta)^2}\,, \\
\dot{p_\theta} &= -\frac{\partial H}{\partial\, \theta} = \frac{p_\phi^2\, \cos\theta}{m\, b^2\, (\sin\theta)^3} - m\, g\, b\, \sin\theta\,, \\
\dot{p_\phi} &= -\frac{\partial H}{\partial\, \phi} = 0 \quad \Rightarrow \quad p_\phi = \text{constant (angular momentum conservation)}\,.
\end{aligned}$$