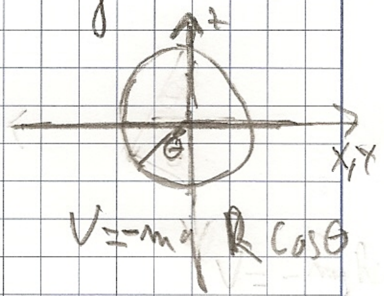
In spherical coordinates with $r = R = \text{const.}$

$$L = \frac{m}{2} R^2 \left(\sin^2(\theta)\, \dot{\phi}^2 + \dot{\theta}^2 \right) + mgR\cos\theta$$



Euler – Lagrange equations:

$$\phi: \quad \frac{d}{dt}\, mR^2 \dot{\phi} \sin^2\theta = 0$$

$$\Rightarrow \quad \underline{\underline{\dot{\phi} \sin^2\theta = \text{const.}}}$$

(Angular momentum conservation)

$$\theta: \quad mR^2 \ddot{\theta} = mR^2 \sin(\theta)\cos(\theta)\, \dot{\phi}^2 - mgR\sin(\theta)$$

Special solution: $\theta = \theta_0 = \text{const.} \Rightarrow$

$$\dot{\phi} = \omega_0 = \text{const} \Rightarrow \underline{\underline{\phi = \omega_0 t + \phi_0}}$$

and $\ddot{\theta} = 0 \Rightarrow$ $(\theta_0 \neq 0, \pi)$ $\underline{\underline{\omega_0^2 = \dfrac{g}{R\cos\theta_0}}}$

Note: $\omega_0^2 > 0 \Rightarrow \theta_0 < \dfrac{\pi}{2}$.

(Note the unconventional definition of θ)