Inductor (L), Capacitor (C), Resistor (R) Circuit (LCR) Notes

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Chapter 1

1.1 LCR circuit

We consider a resistor (R), a capacitor (C) and an inductor (L) in series, see figure 1.1. Charge conservation implies

$$I_R \;=\; I_C \;=\; I_L \;= I \tag{1.1}$$

where I_R, I_C and I_L are the currents at the resistor, capacitor and inductor, respectively.

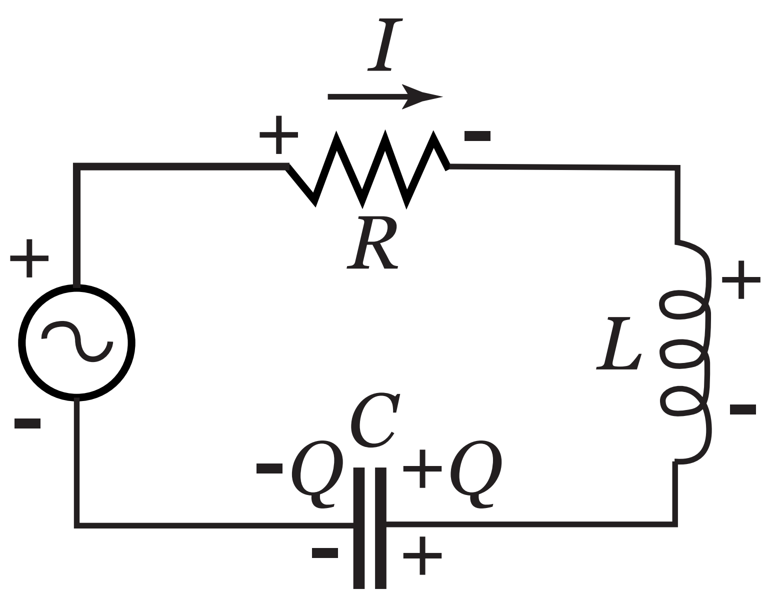

Figure 1.1: Series LCR circuit with an ac generator.

The voltages are related to the currents by the following equations

$$V_R \;=\; R\,I \;\;\text{and}\;\; V_C \;=\; \frac{Q}{C} \;\;\text{with}\;\; \frac{dQ}{dt} \;=\; I \tag{1.2}$$

where $Q(t)$ is the charge on the capacitor. As inductor we consider an idealized solenoid (i.e., we neglect boundary effects which distinguish a real solenoid from an infinitely long one). The induced (back) electromotive force (voltage over the inductor) is the derivative of the magnetic flux

$$\epsilon_{\rm back} \;=\; V_L \;=\; \frac{1}{c}\frac{d\Phi_m}{dt} \;=\; L\,\frac{dI}{dt} \tag{1.3}$$

where the constant $L > 0$ is called *self inductance* of the coil. In accordance with Lenz's law the induced electromotive $\epsilon_{\rm back}$ will work against the electromotive force $\epsilon_{\rm emf}$ which drives the circuit. We use complex calculus and the physical quantities are given by the real parts. For the total voltage we have

$$\epsilon_{\rm emf} \;=\; \epsilon_{\rm max}\, e^{-i\omega t} \;=\; V \;=\; V_L + V_R + V_C \tag{1.4}$$

where ω is the driving frequency. We replace in this relation the voltages by their equations listed above and eliminate the current I in favor of the time derivative of the charge of the capacitor and to arrive at the second order differential equation ($-i\omega\, t$ is Jackson convention)

$$L\,\frac{d^2Q}{dt^2} + R\,\frac{dQ}{dt} + \frac{Q}{C} \;=\; \epsilon_{\rm max}\, e^{-i\omega t}\ . \tag{1.5}$$

The general solution of this equation is obtained as superposition of a special solution of the inhomogenous equation and the general solution of the homogeneous equation, $Q = Q_h + Q_i$. A special solution of the inhomogeneous equation is

$$Q_i \;=\; Q_0\, e^{-i\omega t} \tag{1.6}$$

where

$$\left(-\omega^2\, L - i\,\omega\, R + \frac{1}{C}\right) Q_0 = \epsilon_{\rm max} \quad\Rightarrow\quad Q_0 = \frac{\epsilon_{\rm max}}{-\omega^2\, L - i\,\omega\, R + 1/C}\ . \tag{1.7}$$

The general solution of the homogeneous equation follows from the ansatz $Q_h \sim e^{i\alpha t}$, which implies the algebraic equation

$$-\alpha^2\, L + i\,\alpha\, R + C^{-1} \;=\; 0\ .$$

The solution is

$$\alpha \;=\; i\,\frac{R}{2L} \pm \omega_h \ \text{ with } \ \omega_h = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \tag{1.8}$$

and we arrive at the general solution of the homogeneous equation

$$Q_h \;=\; A_1\, e^{-tR/(2L)}\, e^{+i\omega_h\, t} + A_2\, e^{-tR/(2L)}\, e^{-i\omega_h\, t}\ . \tag{1.9}$$

Due to the exponential damping the homogeneous solution dies out for large times and we are left with the inhomogeneous solution $Q = Q_i$ (1.6). The homogeneous solution has to be used to get the initial values right, e.g. for turning the circuit on or off. In the large time limit the essential features of the LCR circuit are those of the inhomogeneous solution, which we investigate now in more details.

For the inhomogeneous solution (1.6) Q_0 is a complex constant. To separate the amplitude (magnitude) of Q_0 from its phase we write

$$Q_0 = Q_{\max}\, e^{-i\delta} \;\text{ with }\; Q_{\max} = |Q_0| = \frac{\epsilon_{\max}}{\sqrt{\left(1/C - \omega^2 L\right)^2 + \omega^2 R^2}} \,. \tag{1.10}$$

Following standard conventions, we rewrite the expression for $Q_{\max}$ as

$$Q_{\max} = \frac{\epsilon_{\max}}{\omega\sqrt{\left[1/(\omega\, C) - \omega\, L\right]^2 + R^2}} = \frac{\epsilon_{\max}}{\omega\sqrt{(X_C - X_L)^2 + R^2}} = \frac{\epsilon_{\max}}{\omega\, Z} \,. \tag{1.11}$$

Here

$$X_C = \frac{1}{\omega\, C} \;\text{ is called } \textit{capacitive reactance}, \tag{1.12}$$

$$X_L = \omega\, L \;\text{ is called } \textit{inductive reactance}, \tag{1.13}$$

and the difference $X_L - X_C$ is called *total reactance*. The quantity

$$Z = \sqrt{(X_C - X_L)^2 + R^2} \;\text{ is called } \textit{impedance}. \tag{1.14}$$

Using this notation the original equation (1.7) for Q_0 becomes

$$Q_0 = \frac{\epsilon_{\max}}{\omega\, (X_C - X_L - iR)} = \frac{\epsilon_{\max}\,(X_C - X_L + i\, R)}{\omega\, Z^2} \,. \tag{1.15}$$

It follows from $Q_0 = Q_{\max}\, \cos(\delta) - i\, Q_{\max}\, \sin(\delta)$ that the phase δ is given by[1]

$$\tan\delta = -\frac{\sin(\delta)}{\cos(\delta)} = -\frac{\operatorname{Im} Q_0}{\operatorname{Re} Q_0} = \frac{R}{X_L - X_C} \,. \tag{1.16}$$

From equation (1.2) we see that V_C is in phase with the charge Q

$$V_C = \frac{Q_{\max}}{C}\, e^{-i\delta}\, e^{-i\omega t} \tag{1.17}$$

and the phase of the current I is shifted by $\pi/2$ with respect to V_C:

$$\begin{aligned} I = \frac{dQ}{dt} = -i\,\omega\, Q_{\max}\, e^{-i\delta}\, e^{-i\omega t} &= \omega\, Q_{\max}\, e^{-i\delta - i\pi/2}\, e^{-i\omega t} \\ &= I_{\max}\, e^{-i\delta - i\pi/2}\, e^{-i\omega t} \end{aligned} \tag{1.18}$$

where we used $-i = \exp(-i\pi/2)$. Note also that $I_{\max} = \omega\, Q_{\max}$ together with equation (1.10) implies

$$I_{\max} = \frac{\epsilon_{\max}}{Z} = \frac{\epsilon_{\max}}{\sqrt{\left[1/(\omega\, C) - \omega\, L\right]^2 + R^2}} \,. \tag{1.19}$$

[1] Note that in Tipler, Physics 1995 edition, the phase δ is defined with respect to the current instead of the charge, what leads to the equation $\delta = (X_L - X_C)/R$.

The voltage over the resistor is in phase with the current

$$V_R \;=\; \frac{I_{\rm max}}{R}\, e^{-i\delta - i\pi/2}\, e^{-i\omega t} \tag{1.20}$$

whereas the voltage over the inductor is shifted by another $\pi/2$ phase factor:

$$V_L \;=\; L\,\frac{dI}{dt} \;=\; \omega\, L\, I_{\rm max}\, e^{-i\delta t - i\pi}\, e^{i\omega t}\ . \tag{1.21}$$

The current amplitude (1.19) takes its maximum at the *resonance frequency*

$$\omega_0 \;=\; \frac{1}{\sqrt{LC}}\ . \tag{1.22}$$

Note that $Q_{\rm max}(\omega_0) = \epsilon_{\rm max}/(\omega_0 R)$ due to equation (1.11). For $R \neq 0$ the maximum of $Q_{\rm max}$ is shifted by an amount of order $(R/L)^2$ away from ω_0. We find it by differentiation of $Q_{\rm max}$ with respect to ω^2:

$$-2\,L\,(C^{-1} - \omega^2_{\rm max}\,L) + R^2 = 0 \quad\Longrightarrow\quad \omega_{\rm max} = \sqrt{\frac{1}{LC} - \frac{R^2}{2\,L^2}}\ . \tag{1.23}$$

Note, that this $\omega_{\rm max}$ value is lower than the frequency ω_h of the damped homogeneous solution (1.9), while the resonance frequency ω_0 is larger.