ADVANCED DYNAMICS — PHY 4241/5227
MOMENTUM CONSERVATION

Linear Momentum

In accordance with Noether's theorem, we derive momentum conservation from translation invariance

$$\delta_x L = 0\,, \qquad \delta_y L = 0 \qquad \text{and} \qquad \delta_z L = 0\,. \tag{1}$$

Using the definitions of the variations and the Euler-Lagrange equations, we find for a n particle system (note that the displacements agree for all particles)

$$0 = \delta_x L = \sum_{i=1}^{n} \frac{\partial L}{\partial x_i}\,\delta x \;\Leftrightarrow\; 0 = \sum_{i=1}^{n} \frac{\partial L}{\partial x_i} = \sum_{i=1}^{n} \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = \sum_{i=1}^{n} \dot{p}_i^x \;\Leftrightarrow\; \sum_{i=1}^{n} p_i^x = \text{constant}\,, \tag{2}$$

$$0 = \delta_y L = \sum_{i=1}^{n} \frac{\partial L}{\partial y_i}\,\delta y \;\Leftrightarrow\; 0 = \sum_{i=1}^{n} \frac{\partial L}{\partial y_i} = \sum_{i=1}^{n} \frac{d}{dt}\frac{\partial L}{\partial \dot{y}_i} = \sum_{i=1}^{n} \dot{p}_i^y \;\Leftrightarrow\; \sum_{i=1}^{n} p_i^y = \text{constant}\,, \tag{3}$$

$$0 = \delta_z L = \sum_{i=1}^{n} \frac{\partial L}{\partial z_i}\,\delta z \;\Leftrightarrow\; 0 = \sum_{i=1}^{n} \frac{\partial L}{\partial z_i} = \sum_{i=1}^{n} \frac{d}{dt}\frac{\partial L}{\partial \dot{z}_i} = \sum_{i=1}^{n} \dot{p}_i^z \;\Leftrightarrow\; \sum_{i=1}^{n} p_i^z = \text{constant}\,. \tag{4}$$

In vector notation the result reads

$$\vec{P} = \sum_{i=1}^{n} \vec{p} = \text{constant}\,. \tag{5}$$

Angular Momentum

The total angular momentum of a system is

$$\vec{L} = \sum_j \vec{r}_j \times \vec{p}_j \tag{6}$$

where the sum of over point particles. We now consider an infinitesimal rotations by an angle ϕ:

$$\delta\vec{r}_j = \delta\vec{\phi} \times \vec{r}_j\,, \quad \delta\dot{\vec{r}}_j = \delta\vec{\phi} \times \dot{\vec{r}}_j\,. \tag{7}$$

For example, if this rotation is about the z axis $|\delta\vec{r}_j| = |\delta\phi\, r_j \,\sin(\theta)|$ holds. The components of the rotations (7) are

$$\delta x_j^i \qquad \text{and} \qquad \delta\dot{x}_j^i\,, \tag{8}$$

where $i = 1, 2, 3$ labels the coordinates and $j = 1, \dots, n$ the particles. Assuming symmetry of the Lagrangian under the rotation, we have

$$0 = \sum_j \left\{ \sum_i \frac{\partial L}{\partial x_j^i}\,\delta x_j^i + \sum_i \frac{\partial L}{\partial \dot{x}_j^i}\,\delta\dot{x}_j^i \right\} \tag{9}$$

Using the definition of the genralized momentum and Euler-Lagrange equations, this reads

$$0 \;=\; \sum_j \left\{ \sum_i \dot{p}_j^i \, \delta x_j^i + \sum_i p_j^i \, \delta \dot{x}_j^i \right\} \;=\; \sum_j \left\{ \dot{\vec{p}}_j \cdot \delta \vec{r}_j + \vec{p}_j \cdot \delta \dot{\vec{r}}_j \right\} . \qquad (10)$$

Inserting the definitions (7)

$$0 \;=\; \sum_j \left\{ \dot{\vec{p}}_j \cdot (\delta\vec{\phi} \times \vec{r}_j) + \vec{p}_j \cdot (\delta\vec{\phi} \times \dot{\vec{r}}_j) \right\} \qquad (11)$$

and we want to pull out the $\delta\vec{\phi}$ rotations as they are the same for all particles:

$$0 \;=\; \sum_j \left\{ \delta\vec{\phi} \cdot (\vec{r}_j \times \dot{p})j) + \delta\vec{\phi} \cdot (\dot{\vec{r}}_j \times \vec{p}_j) \right\} \;=\; \delta\vec{\phi} \, \frac{d}{dt} \sum_j (\vec{r}_j \times \vec{p}_j) \qquad (12)$$

$$\Leftrightarrow \; \sum_j (\vec{r}_j \times \vec{p}_j) \;=\; \vec{L} \;=\; \text{Constant} \, . \qquad (13)$$