PROBLEM 31

The Matrix L is defined by:

$$L = \begin{pmatrix} l^0{}_0 & l^0{}_1 & l^0{}_2 & l^0{}_3 \\ l^1{}_0 & l^1{}_1 & l^1{}_2 & l^1{}_3 \\ l^2{}_0 & l^2{}_1 & l^2{}_2 & l^2{}_3 \\ l^3{}_0 & l^3{}_1 & l^3{}_2 & l^3{}_3 \end{pmatrix}.$$

(a)–Calculate $-gL$. (b)–Write down the transpose matrix $\widetilde{L}$. (c)–Calculate$\widetilde{L}g$. (d)–Compare (a) and (c) to find the general form of L (*i.e.* use $\widetilde{L}g = -gL$).

(a)–The matrix g is defined by:

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

–By matrix multiplication:

$$-gL = \begin{pmatrix} -l^0{}_0 & -l^0{}_1 & -l^0{}_2 & -l^0{}_3 \\ l^1{}_0 & l^1{}_1 & l^1{}_2 & l^1{}_3 \\ l^2{}_0 & l^2{}_1 & l^2{}_2 & l^2{}_3 \\ l^3{}_0 & l^3{}_1 & l^3{}_2 & l^3{}_3 \end{pmatrix}.$$

(b)–The transpose of L is:

$$\widetilde{L} = \begin{pmatrix} l^0{}_0 & l^1{}_0 & l^2{}_0 & l^3{}_0 \\ l^0{}_1 & l^1{}_1 & l^2{}_1 & l^3{}_1 \\ l^0{}_2 & l^1{}_2 & l^2{}_2 & l^3{}_2 \\ l^0{}_3 & l^1{}_3 & l^2{}_3 & l^3{}_3 \end{pmatrix}.$$

(c)–By matrix multiplication:

$$\widetilde{L}g = \begin{pmatrix} l^0{}_0 & -l^1{}_0 & -l^2{}_0 & -l^3{}_0 \\ l^0{}_1 & -l^1{}_1 & -l^2{}_1 & -l^3{}_1 \\ l^0{}_2 & -l^1{}_2 & -l^2{}_2 & -l^3{}_2 \\ l^0{}_3 & -l^1{}_3 & -l^2{}_3 & -l^3{}_3 \end{pmatrix}.$$

(d)–Set $-gL = \widetilde{L}g$ and compare the components:

$$\begin{pmatrix} -l^0{}_0 = l^0{}_0 & -l^0{}_1 = -l^1{}_0 & -l^0{}_2 = -l^2{}_0 & -l^0{}_3 = -l^3{}_0 \\ l^1{}_0 = l^0{}_1 & l^1{}_1 = -l^1{}_1 & l^1{}_2 = -l^2{}_1 & l^1{}_3 = -l^3{}_1 \\ l^2{}_0 = l^0{}_2 & l^2{}_1 = -l^1{}_2 & l^2{}_2 = -l^2{}_2 & l^2{}_3 = -l^3{}_2 \\ l^3{}_0 = l^0{}_3 & l^3{}_1 = -l^1{}_3 & l^3{}_2 = -l^2{}_3 & l^3{}_3 = -l^3{}_3 \end{pmatrix} . \tag{1}$$

–The resulting matrix is:

$$L = \begin{pmatrix} 0 & l^0{}_1 & l^0{}_2 & l^0{}_3 \\ l^0{}_1 & 0 & l^1{}_2 & l^1{}_3 \\ l^0{}_2 & -l^1{}_2 & 0 & l^2{}_3 \\ l^0{}_3 & -l^1{}_3 & -l^2{}_3 & 0 \end{pmatrix} . \tag{2}$$

(e)–Obtain the same result by discussing the elements of $g^{\alpha\beta}\,\tilde{l}_\beta{}^\gamma\, g_{\gamma\delta} = -l^\alpha{}_\delta$. Performing the contractions we get

$$\tilde{l}^\alpha{}_\delta = -l^\alpha{}_\delta \, .$$

Using the definition

$$\tilde{l}^\alpha{}_\delta = l_\delta{}^\alpha$$

we have the equations

$$l_\delta{}^\alpha = -l^\alpha{}_\delta \, ,$$

which are identical with the equations (1), namely (no summations, $a = 0,1,2,3$, $i = 1,2,3$ and $j = 1,2,3$):

$$l_a{}^a = +l^a{}_a \Rightarrow l^a{}_a = -l^a{}_a = 0 \, ,$$

$$l_0{}^i = -l^0{}_i \Rightarrow l^0{}_i = +l^i{}_0 \, ,$$

$$l_j{}^i = +l^j{}_i \Rightarrow l^j{}_i = -l^i{}_j$$

and the resulting matrix is again (2).