

**ADVANCED DYNAMICS — PHY 4241/5227**  
**HOME AND CLASS WORK – SET 2**

Solution for assignment 10: Double Pendulum.

Derivation of the Lagrangian:

$$\begin{aligned}x_1 &= l \sin \phi \\y_1 &= l \cos \phi \\x_2 &= l \sin \phi + l \sin \psi \\y_2 &= l \cos \phi + l \cos \psi \\L &= \frac{1}{2} m l \left( \dot{x}_1^2 + \dot{y}_1^2 \right) + \frac{1}{2} m l \left( \dot{x}_2^2 + \dot{y}_2^2 \right) - m g l (z_1 + z_2) + \text{const} \\&= \frac{1}{2} m l \dot{\phi}^2 + \frac{1}{2} m l \left( \dot{\phi}^2 + \dot{\psi}^2 + 2 \dot{\phi} \dot{\psi} \cos(\phi - \psi) \right) + m g l (2 \cos \phi + \cos \psi) \\&= m l \left( \dot{\phi}^2 + \frac{1}{2} \dot{\psi}^2 + \dot{\phi} \dot{\psi} \cos(\phi - \psi) \right) + m g l (2 \cos \phi + \cos \psi) .\end{aligned}$$

Taylor expansion to second order in  $\phi$ ,  $\psi$ ,  $\dot{\phi}$  and  $\dot{\psi}$  gives

$$L = m l \left( \dot{\phi}^2 + \frac{1}{2} \dot{\psi}^2 + \dot{\phi} \dot{\psi} \right) - m g l \left( \phi^2 + \frac{1}{2} \psi^2 \right) + 3 m g l .$$

The Euler-Lagrange equation are then

$$\begin{aligned}0 &= \frac{d}{dt} \frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi} = 2 \ddot{\phi} + \ddot{\psi} + 2 \frac{g}{l} \phi \\0 &= \frac{d}{dt} \frac{\partial L}{\partial \dot{\psi}} - \frac{\partial L}{\partial \psi} = \ddot{\psi} + \ddot{\phi} + \frac{g}{l} \psi\end{aligned}$$

and in matrix notation

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \ddot{\phi} \\ \ddot{\psi} \end{pmatrix} + \begin{pmatrix} 2g/l & 0 \\ 0 & g/l \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix} = 0 .$$