Solution for assignment 10c: Double pendulum eigenvectors and explicit orbits.

$$\frac{g}{l} \begin{pmatrix} -2\left(2 \pm \sqrt{2}\right) + 2 & -\left(2 \pm \sqrt{2}\right) \\ -\left(2 \pm \sqrt{2}\right) & -\left(2 \pm \sqrt{2}\right) + 1 \end{pmatrix} \begin{pmatrix} \phi_0 \\ \psi_0 \end{pmatrix} = \frac{g}{l} \begin{pmatrix} a_\pm & b_\pm \\ b_\pm & c_\pm \end{pmatrix} \begin{pmatrix} \phi_0 \\ \psi_0 \end{pmatrix} = 0$$

with $a_+ = -2 - 2\sqrt{2}$, $b_+ = -2 - \sqrt{2}$, $c_+ = -1 - \sqrt{2}$ and $a_- = -2 + 2\sqrt{2}$, $b_- = -2 + \sqrt{2}$, $c_- = -1 + \sqrt{2}$. With the ansatz given in the problem we find:

$$\begin{pmatrix} a_\pm & b_\pm \\ b_\pm & c_\pm \end{pmatrix} \begin{pmatrix} 1 \\ \psi_{0\pm} \end{pmatrix} = 0 \quad \Rightarrow \quad \psi_{0\pm} = -\frac{a_\pm}{b_\pm} = -\frac{b_\pm}{c_\pm}, \quad \psi_{0+} = -\sqrt{2} \ \text{ and } \ \psi_{0-} = \sqrt{2}$$

and, therefore, the eigenvectors

$$\Phi_{0+} = \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} \quad \text{and} \quad \Phi_{0-} = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}.$$

They are **not** orthogonal, because the previously solved eigenvalue problem is of the form

$$V\,\Phi_0 = \omega^2\, T\, \Phi_0 \quad \text{with} \quad V = \begin{pmatrix} 2g/l & 0 \\ 0 & g/l \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Multiplying with T^{-1} from the left does not help, because the matrix $T^{-1}V$ will not be symmetric. Instead a generalization of the usual orthogonality definition works, see M&T (12.55) or Goldstein Mechanics. The general solution for the double pendulum problem is given by the real part of our ansatz, which can be written

$$\Phi(t) = \left(A_+ \cos \omega_+ t + B_+ \sin \omega_+ t\right) \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} + \left(A_- \cos \omega_- t + B_- \sin \omega_- t\right) \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}.$$

The four constants are determined by the four initial value, e.g., ϕ_0, $\dot{\phi}_0$, ψ_0, $\dot{\psi}_0$ at time $t = 0$:

$$\begin{aligned} \phi_0 &= A_+ + A_-, & \psi_0 &= \sqrt{2}\,(-A_+ + A_-), \\ \dot{\phi}_0 &= \omega_+ B_+ + \omega_- B_-, & \dot{\psi}_0 &= \sqrt{2}\,(-\omega_+ B_+ + \omega_- B_-), \end{aligned}$$

which gives

$$\begin{aligned} A_+ &= \frac{\phi_0}{2} - \frac{\psi_0}{2\sqrt{2}}, & A_- &= \frac{\phi_0}{2} + \frac{\psi_0}{2\sqrt{2}}, \\ B_+ &= \frac{\dot{\phi}_0}{2\omega_+} - \frac{\dot{\psi}_0}{2\sqrt{2}\,\omega_+}, & B_- &= \frac{\dot{\phi}_0}{2\omega_-} + \frac{\dot{\psi}_0}{2\sqrt{2}\,\omega_-}, \end{aligned}$$

and the time dependence of the angles is

$$\begin{aligned} \phi(t) &= +A_+ \cos \omega_+ t + B_+ \sin \omega_+ t + A_- \cos \omega_- t + B_- \sin \omega_- t, \\ \psi(t) &= \left(-A_+ \cos \omega_+ t - B_+ \sin \omega_+ t + A_- \cos \omega_- t + B_- \sin \omega_- t\right)\sqrt{2}. \end{aligned}$$

For the initial conditions

$$\phi_0 = 0, \quad \dot{\phi}_0 = 1, \quad \psi_0 = 0, \quad \dot{\psi}_0 = -1$$

at time $t = 0$ see the figure for a plot up to $t = 50\,\sqrt{l/g}$ (next page).