ADVANCED DYNAMICS — PHY 4241/5227

HOME AND CLASS WORK – SET 7

Solution for assignment 10c: Double pendulum eigenvectors and explicit orbits.

$$\frac{g}{l}\begin{pmatrix} -2\,(2\pm\sqrt{2})+2 & -(2\pm\sqrt{2}) \\ -(2\pm\sqrt{2}) & -(2\pm\sqrt{2})+1 \end{pmatrix}\begin{pmatrix}\phi_0\\ \psi_0\end{pmatrix} \;=\; \frac{g}{l}\begin{pmatrix} a_\pm & b_\pm \\ b_\pm & c_\pm\end{pmatrix}\begin{pmatrix}\phi_0\\ \psi_0\end{pmatrix} \;=\; 0$$

with $a_+ = -2-2\sqrt{2}$, $b_+ = -2-\sqrt{2}$, $c_+ = -1-\sqrt{2}$ and $a_- = -2+2\sqrt{2}$, $b_- = -2+\sqrt{2}$, $c_- = -1+\sqrt{2}$. With the ansatz given in the problem we find:

$$\begin{pmatrix} a_\pm & b_\pm \\ b_\pm & c_\pm\end{pmatrix}\begin{pmatrix}1\\ \psi_{0\pm}\end{pmatrix} = 0 \;\;\Rightarrow\;\; \psi_{0\pm} = -\frac{a_\pm}{b_\pm} = -\frac{b_\pm}{c_\pm}\,,\quad \psi_{0+} = -\sqrt{2}\;\text{ and }\;\psi_{0-} = \sqrt{2}$$

and, therefore, the eigenvectors

$$\Phi_{0+} = \begin{pmatrix}1\\ -\sqrt{2}\end{pmatrix}\quad\text{and}\quad \Phi_{0-} = \begin{pmatrix}1\\ \sqrt{2}\end{pmatrix}\,.$$

They are **not** orthogonal, because the previously solved eigenvalue problem is of the form

$$V\,\Phi_0 = \omega^2\,T\,\Phi_0 \quad\text{with}\quad V = \begin{pmatrix} 2g/l & 0 \\ 0 & g/l\end{pmatrix}\;\text{ and }\; T = \begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}\,.$$

Multiplying with T^{-1} from the left does not help, because the matrix $T^{-1}V$ will not be symmetric. Instead a generalization of the usual orthogonality definition works, see M&T (12.55) or Goldstein Mechanics. The general solution for the double pendulum problem is given by the real part of our ansatz, which can be written

$$\Phi(t) = (A_+\cos\omega_+ t + B_+\sin\omega_+ t)\begin{pmatrix}1\\ -\sqrt{2}\end{pmatrix} + (A_-\cos\omega_- t + B_-\sin\omega_- t)\begin{pmatrix}1\\ \sqrt{2}\end{pmatrix}\,.$$

The four constants are determined by the four initial value, e.g., ϕ_0, $\dot\phi_0$, ψ_0, $\dot\psi_0$ at time $t=0$:

$$\begin{aligned} \phi_0 &= A_+ + A_-\,, & \psi_0 &= \sqrt{2}\,(-A_+ + A_-)\,,\\ \dot\phi_0 &= \omega_+ B_+ + \omega_- B_-\,, & \dot\psi_0 &= \sqrt{2}\,(-\omega_+ B_+ + \omega_- B_-)\,, \end{aligned}$$

which gives

$$\begin{aligned} A_+ &= \frac{\phi_0}{2} - \frac{\psi_0}{2\sqrt{2}}\,, & A_- &= \frac{\phi_0}{2} + \frac{\psi_0}{2\sqrt{2}}\,,\\ B_+ &= \frac{\dot\phi_0}{2\omega_+} - \frac{\dot\psi_0}{2\sqrt{2}\,\omega_+}\,, & B_- &= \frac{\dot\phi_0}{2\omega_-} + \frac{\dot\psi_0}{2\sqrt{2}\,\omega_-}\,, \end{aligned}$$

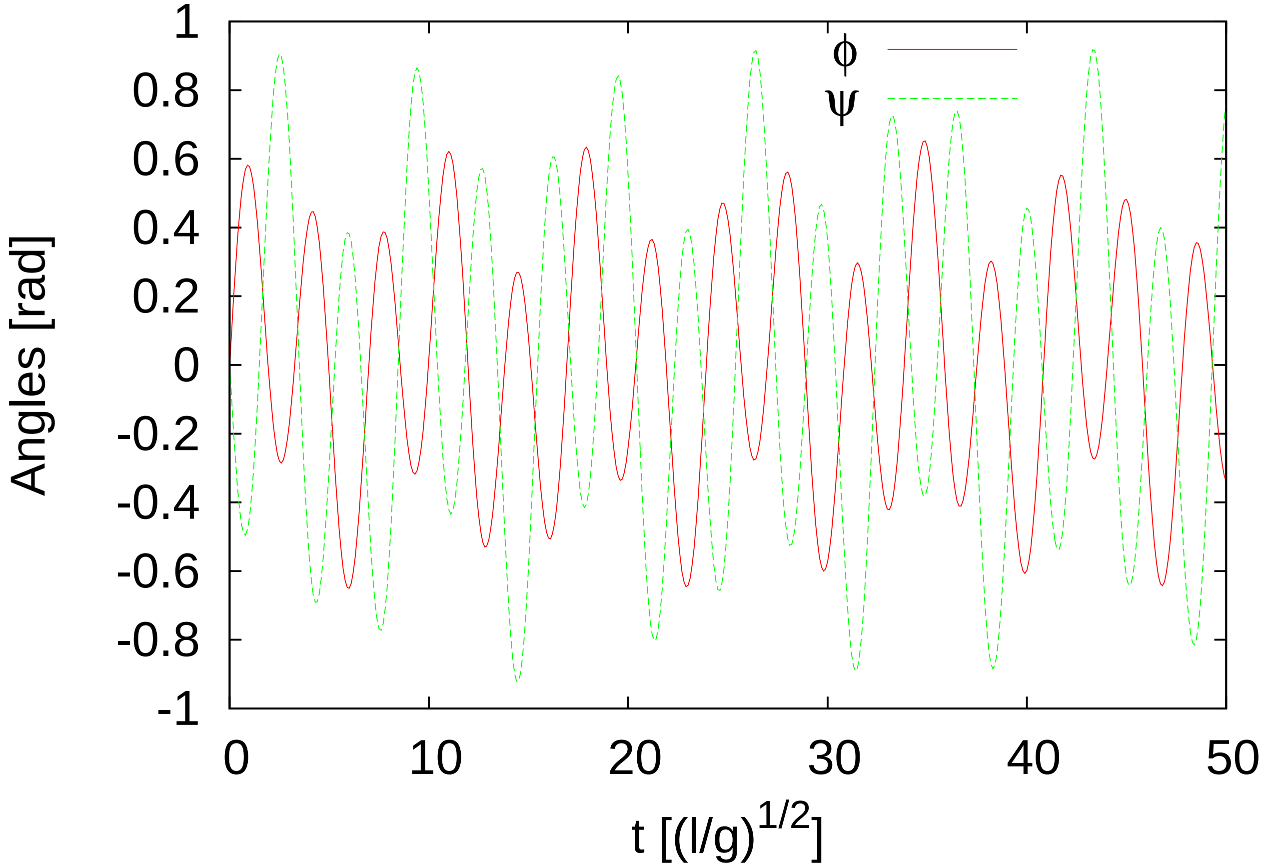
and the time dependence of the angles is

$$\begin{aligned} \phi(t) &= +A_+\cos\omega_+ t + B_+\sin\omega_+ t + A_-\cos\omega_- t + B_-\sin\omega_- t\,,\\ \psi(t) &= (-A_+\cos\omega_+ t - B_+\sin\omega_+ t + A_-\cos\omega_- t + B_-\sin\omega_- t)\,\sqrt{2}\,. \end{aligned}$$

For the initial conditions

$$\phi_0 = 0\,,\quad \dot\phi_0 = 1\,,\quad \psi_0 = 0\,,\quad \dot\psi_0 = -1$$

at time $t=0$ see the figure for a plot up to $t = 50\sqrt{l/g}$ (next page).



Angles [rad]
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
10
20
30
40
50
t [(l/g)^{1/2}]
φ
ψ