Solution for assignment 14 set 4

Poisson Brackets

Let us consider functions $g = g(q_k, p_k, t)$ and $h = h(q_k, p_k, t)$. The Poisson bracket is defined by

$$[g, h] \stackrel{\text{def}}{=} \sum_k \left(\frac{\partial g}{\partial q_k} \frac{\partial h}{\partial p_k} - \frac{\partial h}{\partial q_k} \frac{\partial g}{\partial p_k} \right) .$$

The properties of the assignment are shown in the following:

1. Resulting in the definition of the total time derivative:

$$\begin{aligned} \frac{dg}{dt} = [g, H] + \frac{\partial g}{\partial t} &= \sum_k \left(\frac{\partial g}{\partial q_k} \frac{\partial H}{\partial p_k} - \frac{\partial H}{\partial q_k} \frac{\partial g}{\partial p_k} \right) + \frac{\partial g}{\partial t} \\ &= \sum_k \left(\frac{\partial g}{\partial q_k} \dot{q}_k + \frac{\partial g}{\partial p_k} \dot{p}_k \right) + \frac{\partial g}{\partial t} . \end{aligned}$$

2. Using $\partial q_j / \partial p_k = 0$:

$$\dot{q}_j = [q_j, H] = \sum_k \left(\frac{\partial q_j}{\partial q_k} \frac{\partial H}{\partial p_k} - \frac{\partial H}{\partial q_k} \frac{\partial q_j}{\partial p_k} \right) = \sum_k \delta_{jk} \, \dot{q}_k = \dot{q}_j ,$$

3. Using $\partial p_j / \partial q_k = 0$:

$$\dot{p}_j = [p_j, H] = \sum_k \left(\frac{\partial p_j}{\partial q_k} \frac{\partial H}{\partial p_k} - \frac{\partial H}{\partial q_k} \frac{\partial p_j}{\partial p_k} \right) = \sum_k \delta_{jk} \, \dot{p}_k = \dot{p}_j ,$$

4. Commutators of generalized coordinates and momenta:

$$[q_i, q_j] = \sum_k \left(\frac{\partial q_i}{\partial q_k} \frac{\partial q_j}{\partial p_k} - \frac{\partial q_j}{\partial q_k} \frac{\partial q_i}{\partial p_k} \right) = 0$$

as $\partial q_j / \partial p_k = 0$ and $\partial q_i / \partial p_k = 0$. Similarly

$$[p_i, p_j] = \sum_k \left(\frac{\partial p_i}{\partial q_k} \frac{\partial p_j}{\partial p_k} - \frac{\partial p_j}{\partial q_k} \frac{\partial p_i}{\partial p_k} \right) = 0 ,$$

while

$$[q_i, p_j] = \sum_k \left(\frac{\partial q_i}{\partial q_k} \frac{\partial p_j}{\partial p_k} - \frac{\partial p_j}{\partial q_k} \frac{\partial q_i}{\partial p_k} \right) = \sum_k (\delta_{ik} \, \delta_{jk} - 0) = \delta_{ij} .$$

5. Angular momentum commutator:

$$\begin{aligned} [x_i, L_j] &= \sum_k \sum_l \sum_m \epsilon_{jlm} \left(\frac{\partial x_i}{\partial x_k} \frac{\partial (x_l p_m)}{\partial p_k} - \frac{\partial (x_l p_m)}{\partial x_k} \frac{\partial x_i}{\partial p_k} \right) \\ &= \sum_k \sum_l \sum_m \epsilon_{jlm} \, \delta_{ik} \, x_l \, \delta_{mk} = \sum_l \sum_m \epsilon_{jlm} \, \delta_{im} \, x_l = \sum_l \epsilon_{jli} \, x_l = \sum_l \epsilon_{ijl} \, x_l . \end{aligned}$$

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\begin{aligned}
[p_i, L_j] &= \sum_k \sum_l \sum_m \epsilon_{jlm} \left(\frac{\partial p_i}{\partial x_k} \frac{\partial (x_l p_m)}{\partial p_k} - \frac{\partial (x_l p_m)}{\partial x_k} \frac{\partial p_i}{\partial p_k} \right) \\
&= -\sum_k \sum_l \sum_m \epsilon_{jlm} \, \delta_{lk} \, p_m \, \delta_{ik} \;=\; -\sum_l \sum_m \epsilon_{jlm} \, \delta_{li} \, p_m \;=\; -\sum_m \epsilon_{jim} \, p_m \;=\; \sum_m \epsilon_{ijm} \, x_m \, .
\end{aligned}
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$$
\begin{aligned}
[L_i, L_j] &= \sum_k \sum_l \sum_m \sum_n \sum_r \epsilon_{ikl} \epsilon_{jmn} \left(\frac{\partial (x_k p_l)}{\partial x_r} \frac{\partial (x_m p_n)}{\partial p_r} - \frac{\partial (x_m p_n)}{\partial x_r} \frac{\partial (x_k p_l)}{\partial p_r} \right) \\
&= \sum_k \sum_l \sum_m \sum_n \sum_r \epsilon_{ikl} \epsilon_{jmn} \left(\delta_{kr} \delta_{nr} x_m p_l - \delta_{mr} \delta_{lr} x_k p_n \right) \\
&= \sum_k \sum_l \sum_m \sum_n \epsilon_{ikl} \epsilon_{jmn} \left(\delta_{kn} x_m p_l - \delta_{ml} x_k p_n \right) \\
&= \sum_k \sum_l \left(\sum_m \epsilon_{ikl} \epsilon_{jmk} x_m p_l - \sum_n \epsilon_{ikl} \epsilon_{jln} x_k p_n \right) \\
&= \sum_k \sum_l \left(\sum_m \epsilon_{kli} \epsilon_{kjm} x_m p_l - \sum_n \epsilon_{lik} \epsilon_{lnj} x_k p_n \right) \\
&= \sum_l \sum_m (\delta_{lj} \delta_{im} - \delta_{lm} \delta_{ij}) x_m p_l - \sum_k \sum_n (\delta_{in} \delta_{kj} - \delta_{ij} \delta_{kn}) x_k p_n \\
&= x_i p_j - \delta_{ij} \, \vec{x} \cdot \vec{p} - x_j p_i + \delta_{ij} \, \vec{x} \cdot \vec{p} \;=\; x_i p_j - x_j p_i \, .
\end{aligned}
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This agrees with

$$
\begin{aligned}
\sum_k \epsilon_{ijk} L_k &= \sum_k \sum_l \sum_m \epsilon_{ijk} \epsilon_{klm} x_l p_m \;=\; \sum_k \sum_l \sum_m \epsilon_{kij} \epsilon_{klm} x_l p_m \\
&= \sum_l \sum_m (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) \; x_l p_m \;=\; x_i p_j - x_j p_i \, .
\end{aligned}
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