

Solution for assignment 19 set 6.

Rewritten and using $\cos \theta = x/r$, the initial equation becomes

$$p = r(1 + e \cos \theta) = r(1 + e x/r) = r + e x \quad \text{or} \quad r = p - e x.$$

Squaring both sides

$$x^2 + y^2 = p^2 - 2 p e x + e^2 x^2.$$

Bringing all terms with x or y to one side,

$$x^2(1 - e^2) + 2 p e x + y^2 = p^2,$$

$$x^2 + \frac{2 p e}{1 - e^2} x + \frac{y^2}{1 - e^2} = \frac{p^2}{1 - e^2},$$

According to the recipe for completion of the square we substitute

$$x' = x + \frac{p e}{1 - e^2}$$

and obtain

$$x'^2 + \frac{y^2}{1 - e^2} = \frac{p^2}{1 - e^2} + \left(\frac{p e}{1 - e^2} \right)^2 = \frac{p^2(1 - e^2) + p^2 e^2}{(1 - e^2)^2} = \frac{p^2}{(1 - e^2)^2},$$

$$x'^2 \frac{(1 - e^2)^2}{p^2} + y^2 \frac{1 - e^2}{p^2} = 1.$$

With the definitions

$$a = \frac{p}{1 - e^2} \quad \text{and} \quad b = \frac{p}{\sqrt{1 - e^2}}$$

this reads

$$\left(\frac{x'}{a} \right)^2 + \left(\frac{y}{b} \right)^2 = 1.$$