

PHY 4938 HOME AND CLASS WORK – SET 9

(November 26, 2011)

Read Landau-Lifshitz §40 and §42.

(36-A) Legendre transformation: Define the Hamiltonian by

$$H = \left(\sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L \right) \text{ and the generalized momentum by } p_j = \frac{\partial L}{\partial \dot{q}_j} .$$

Show that the Hamiltonian is a function of q_j and p_j only: $H = H(q_j, p_j)$. Then derive Hamilton's equations of motion. Hint: Calculate dH . Due in class (4 points).

(36-B) Problem 1, Landau-Lifshitz §40, p.133. Show first: $H = T + U$. Due in class (4 points).

(36-C) Derive Newton's equation from Hamilton's equation for the Cartesian case. Due in class (2 points).

(37) Poisson brackets are defined by

$$[g, h] = \sum_k \left(\frac{\partial g}{\partial q_k} \frac{\partial h}{\partial p_k} - \frac{\partial h}{\partial q_k} \frac{\partial g}{\partial p_k} \right)$$

where g and h are functions of q_i , p_i and, possibly, t . Show the following properties (due December 7 before class, 10 points):

1.

$$\frac{dg}{dt} = [g, H] + \frac{\partial g}{\partial t} .$$

2.

$$\dot{q}_j = [q_j, H] .$$

3.

$$\dot{p}_j = [p_j, H] .$$

4.

$$[x_i, x_j] = [p_i, p_j] = 0 ; \quad [x_i, p_j] = \delta_{ij} ,$$

5.

$$[x_i, L_j] = \epsilon_{ijk} x_k , \quad [p_i, L_j] = \epsilon_{ijk} p_k , \quad \text{and} \quad [L_i, L_j] = \epsilon_{ijk} L_k , \quad ,$$

where the Einstein summation convention is used and $L_j = \epsilon_{jkl} x_k p_l$ is the j th component of the angular momentum of the system.

(38) Liouville's Theorem: We define the velocity in phase space as a $2n$ -dimensional vector $\vec{v} = (\dot{q}_1, \dots, \dot{q}_n, \dot{p}_1, \dots, \dot{p}_n)$. A large collection of particles can be described by their density in phase space $\rho(q_1, \dots, q_n, p_1, \dots, p_n, t)$. If there are no sources or sinks, we have a conserved current

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\vec{v} \rho) = 0 \quad (1)$$

where $\nabla = \left(\frac{\partial}{\partial q_1}, \dots, \frac{\partial}{\partial q_n}, \frac{\partial}{\partial p_1}, \dots, \frac{\partial}{\partial p_n} \right)$ is the gradient in phase space.

(a) Expand (1) in sums of partial derivatives (you get five terms when you keep coordinates and momenta in separate contributions). (b) Use Hamilton's equations of motion to show that two terms cancel out. (c) Combine the remaining terms to

$$\frac{d\rho}{dt} = 0 \quad (\text{Liouville's Theorem}). \quad (2)$$

Due in class (6 points).