Solution 17 set 4.

1. Yes, the Lagrangian does not depend on the time.

2. Solving for $\vec{r}_1$:

$$M\,\vec{R} = m_1\vec{r}_1 + m_2\vec{r}_2\,, \quad M = m_1 + m_2\,,\ \vec{r} = \vec{r}_1 - \vec{r}_2\,,$$

$$m_1\vec{r}_1 = M\,\vec{R} - m_2\vec{r}_2 = M\,\vec{R} + m_2\vec{r} - m_2\vec{r}_1\,,$$
$$\vec{r}_1 = \vec{R} + \frac{m_2}{M}\,\vec{r},$$
$$m_2\vec{r}_2 = M\,\vec{R} - m_1\vec{r}_1 = M\,\vec{R} - m_1\vec{r} - m_1\vec{r}_2\,,$$
$$\vec{r}_2 = \vec{R} - \frac{m_1}{M}\,\vec{r}.$$

3. In the CM frame $\vec{R} = 0$ and using $\mu = m_1 m_2/(m_1 + m_2)$ we can write

$$\vec{r}_1 = \frac{\mu}{m_1}\,\vec{r}, \qquad \vec{r}_2 = -\frac{\mu}{m_2}\,\vec{r}.$$

Therefore,

$$T_{\rm cm} = \frac{1}{2}\left(\frac{\mu^2}{m_1} + \frac{\mu^2}{m_2}\right)\vec{v}^2 = \frac{1}{2}\,\frac{\mu^2\,(m_1+m_2)}{m_1 m_2}\,\vec{v}^2 = \frac{1}{2}\,\mu\,\vec{v}^2$$

4. Angular momentum:

$$m_1\,\vec{r}_1\times\vec{v}_1 + m_2\,\vec{r}_2\times\vec{v}_2 = m_1\,\frac{\mu}{m_1}\,\vec{r}\times\vec{v}_1 - m_2\,\frac{\mu}{m_2}\,\vec{r}\times\vec{v}_2 = \mu\,\vec{r}\times(\vec{v}_1 - \vec{v}_2) = \mu\,\vec{r}\times\vec{v}\ .$$