Solution for assignment 19: Effective Potential.

$$U_{\text{eff}}(r) = -\frac{\alpha}{r} + \frac{M^2}{2mr^2}$$

with M angular momentum and m reduced mass.
(A) Solve now for $r = r_0$:

$$U'_{\text{eff}}(r) = +\frac{\alpha}{r} - \frac{2M^2}{2mr^3} = 0 \Rightarrow \alpha\, r_0 = \frac{M^2}{m} \Rightarrow r_0 = \frac{M^2}{\alpha\, m}\,.$$

(B) The minimum value of the effective potential:

$$U_{\text{eff}}^{\text{min}} = U_{\text{eff}}(r_0) = -\frac{\alpha^2 m}{M^2} + \frac{\alpha^2 m}{2M^2} = -\frac{\alpha^2 m}{2M^2}\,.$$

(C) Calculation of the turning points for $E < 0$: We are looking of solutions of the quadratic equation

$$E\, r^2 + \alpha\, r - \frac{M^2}{2m} = 0$$

$$r^2 + p\, r - q = 0 \text{ with } p = \frac{\alpha}{E}\,, \quad q = -\frac{M^2}{2m\, E}\,,$$

$$r_{\text{min,max}} = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q} = -\frac{\alpha}{2E} \pm \sqrt{\left(\frac{\alpha}{2E}\right)^2 + \frac{M^2}{2mE}}\,.$$

Recall that E is negative.
(D) When do we have real solutions? The upper bound is $E < 0$, the lower follows from

$$\left(\frac{\alpha}{2E}\right)^2 + \frac{M^2}{2mE} = 0 \Rightarrow \frac{\alpha^2}{4E} + \frac{M^2}{2m} = 0 \Rightarrow E = -\frac{\alpha^2 m}{2M^2}\,.$$

So, we find

$$-\frac{\alpha^2 m}{2M^2} < E < 0$$

and the lower value is the minimum of the effective potential as calculated under (B).