Solution for assignment 20: Kepler Ellipse.

Rewritten and using $\cos\theta = x/r$, the initial equation becomes

$$p = r\,(1 + e\,\cos\theta) = r\,(1 + e\,x/r) = r + e\,x \quad \text{or} \quad r = p - e\,x\,.$$

Squaring both sides

$$x^2 + y^2 = p^2 - 2\,p\,e\,x + e^2\,x^2\,.$$

Bringing all terms with x or y to one side,

$$x^2\,(1 - e^2) + 2\,p\,e\,x + y^2 = p^2\,,$$

$$x^2 + \frac{2\,p\,e}{1 - e^2}\,x + \frac{y^2}{1 - e^2} = \frac{p^2}{1 - e^2}\,,$$

According to the recipe for completion of the square we substitute

$$x' = x + \frac{p\,e}{1 - e^2}$$

and obtain

$$x'^{\,2} + \frac{y^2}{1 - e^2} = \frac{p^2}{1 - e^2} + \left(\frac{p\,e}{1 - e^2}\right)^2 = \frac{p^2\,(1 - e^2) + p^2\,e^2}{(1 - e^2)^2} = \frac{p^2}{(1 - e^2)^2}\,,$$

$$x'^{\,2}\,\frac{(1 - e^2)^2}{p^2} + y^2\,\frac{1 - e^2}{p^2} = 1\,.$$

With the definitions

$$a = \frac{p}{1 - e^2} \quad \text{and} \quad b = \frac{p}{\sqrt{1 - e^2}}$$

this reads

$$\left(\frac{x'}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1\,.$$