Solution for assignment 38

Liouville's Theorem

We consider motion of point particles with n degrees of freedom in **phase space**, which is described by a Hamiltonian

$$H(q_1, \ldots, q_n; p_1, \ldots, p_n)\,.$$

Let $\rho(q_1, \ldots, q_n; p_1, \ldots, p_n; t)$ be the density in phase space and the velocity of the density element is the vector

$$\vec{v} \;=\; (\dot{q}_1, \ldots, \dot{q}_n; \dot{p}_1, \ldots, \dot{p}_n)\,.$$

The gradient is now also defined in phase space ($\hat{q}_i$ and $\hat{p}_i$ are unit vectors):

$$\nabla \;=\; \sum_{i=1}^{n} \left(\hat{q}_i \frac{\partial}{\partial q_i} + \hat{p}_i \frac{\partial}{\partial p_i} \right)\,.$$

The continuity equation reads

$$0 \;=\; \frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} \;=\; \frac{\partial \rho}{\partial t} + \nabla \cdot (\vec{v}\rho)$$

as $\vec{j} = \vec{v}\rho$ holds. Therefore,

$$0 \;=\; \frac{\partial \rho}{\partial t} + \rho\, \nabla \cdot \vec{v} + \vec{v} \cdot \nabla\, \rho \;=\; \frac{\partial \rho}{\partial t} + \rho \sum_{i=1}^{n} \left(\frac{\partial \dot{q}_i}{\partial q_i} + \frac{\partial \dot{p}_i}{\partial p_i} \right) + \sum_{i=1}^{n} \left(\frac{\partial \rho}{\partial q_i}\, \dot{q}_i + \frac{\partial \rho}{\partial p_i}\, \dot{p}_i \right)\,.$$

Using Hamilton's equations we have

$$\frac{\partial \dot{q}_i}{\partial q_i} \;=\; +\frac{\partial}{\partial q_i} \frac{\partial H}{\partial p_i} \quad \text{and} \quad \frac{\partial \dot{p}_i}{\partial p_i} \;=\; -\frac{\partial}{\partial p_i} \frac{\partial H}{\partial q_i}\,.$$

Interchanging the derivative these terms cancel one another ($\nabla \cdot \vec{v} = 0$ in phase space) and we are left with Liouville's theorem:

$$0 \;=\; \frac{\partial \rho}{\partial t} + \sum_{i=1}^{n} \left(\frac{\partial \rho}{\partial q_i}\, \dot{q}_i + \frac{\partial \rho}{\partial p_i}\, \dot{p}_i \right) \;=\; \frac{d\rho}{dt}\,.$$

This is the motion of an incompressible fluid, but in phase space instead of coordinate space.